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# REINFORCED MINDLIN PLATES WITH EXTREMAL STIFFNESS

ROBERT LIPTON†
Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, MA 01609, U.S.A.

and

ALEJANDRO DÍAZ
Department of Mechanical Engineering, Michigan State University, East Lansing, MI 48824, U.S.A.

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**Abstract**—For a prescribed area fraction of stiffeners, we characterize the set of stiffener reinforced Mindlin plates with extremal overall stiffness. The method rests upon the derivation of optimal bounds of the Hashin-Shtrikman type. Our method is distinct from the usual Hashin-Shtrikman approach. We make use of the underlying variational structure behind the Hashin-Shtrikman method to show that the use of a comparison material is redundant. We do this by proceeding directly and express the equilibrium equations in terms of positive definite integral operators. The positivity of the operators is used to obtain a new Hilbert space variational principle for the effective stiffness. The associated bounds are shown to be realized by effective rigidities associated with hierarchical laminar arrangements of stiffeners. © 1997 Elsevier Science Ltd.

## 1. INTRODUCTION

We consider a Reisner-Mindlin plate of thickness $h_1$ reinforced using ribs or stiffeners of thickness $h_2 > h_1$. For the purposes of structural design the problem is to find the distribution of ribs providing the stiffest response to a prescribed load. Problems of this type have been treated extensively in the literature. Since the early 80's it has been known that the optimal reinforcement of Kirchoff plates may include infinitely fine arrangements of ribs, as indicated in the work of Cheng and Olhoff, (1981), Lurie *et al.* (1982), and Olhoff *et al.* (1981). Such optimal layout problems are found to be made well-posed by extending the class of designs to include effective rigidity tensors. It is within this class that a globally optimal design can be found, see Cheng and Olhoff (1981), Lurie *et al.* (1982), and Murat and Tartar (1985). The effective rigidity tensor captures the overall limiting behavior of an optimizing sequence of layouts with increasingly oscillatory arrangements of stiffeners. This tensor may be anisotropic and depends on the local geometry of the stiffeners.

For optimal compliance design it is of key importance to have an explicit characterization for the set of extremal effective rigidity tensors that maximize or minimize sums of strain energies, see Allaire and Kohn (1993), Jog *et al.* (1994) and Díaz *et al.* (1995), and the recent papers of Allaire *et al.* (1995) and Cherkaev and Palais (1995).

In this article we provide such a characterization for stiffener reinforced Reisner-Mindlin plates, see Sections 6 and 7. To obtain the characterization we start by finding explicit bounds on the effective rigidity tensor for periodically reinforced plates, see Section 4.

We remark that the assumption of periodicity is general since any effective rigidity tensor can be approximated arbitrarily well by that associated with a suitable period geometry. Such observations hold generically for perfectly bonded linear elastic and heat conducting materials and can be found in the work of Golden and Papanicolaou (1983)

† Current address: Dept. Mathematics, Tufts University, Medford, MA02155, U.S.A.

(in the context of random media) and DalMaso and Kohn (in preparation), in the context of $G$-convergence.

The effective rigidity tensor for Reisner-Mindlin plates has two components: an effective shear stiffness $D_S^e$ and an effective membrane stiffness $D_B^e$.

For a set $\bar{\kappa}_1, \bar{\kappa}_2, \ldots, \bar{\kappa}_N$ of constant curvatures and constant transverse shears $\bar{\gamma}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_N$ the sum of energies is written

$$\sum_{j=1}^{N} (D_B^e \bar{\kappa}_j : \bar{\kappa}_j + D_S^e \bar{\gamma}_j \cdot \bar{\gamma}_j).$$

The bounds on the effective tensor are given in terms of two geometric parameters characterizing the composite structure. The first is the area fraction of the stiffeners $\theta_2$ and the second is a probability measure on the unit circle $\mu$ describing the anisotropy of the composite.

Similar measures describing the anisotropy of composites have appeared earlier in the context of multi-phase elastic composites, see Willis (1982), and Avellaneda and Milton (1989). For fixed values of $\theta_2$ and $\mu$ the bounds on the effective tensor are given by tensors $(D_B^+, D_S^+)$, $(D_B^-, D_S^-)$ that depend explicitly upon $\theta_2$ and $\mu$, see Sections 4 and 6. For fixed values of $\theta_2$ and $\mu$ we show that the estimate

$$\sum_{j=1}^{N} (D_B^- \bar{\kappa}_j : \bar{\kappa}_j + D_S^- \bar{\gamma}_j \cdot \bar{\gamma}_j) \leqslant \sum_{j=1}^{N} (D_B^e \bar{\kappa}_j : \bar{\kappa}_j + D_S^e \bar{\gamma}_j \cdot \bar{\gamma}_j) \leqslant \sum_{j=1}^{N} (D_B^+ \bar{\kappa}_j : \bar{\kappa}_j + D_S^+ \bar{\gamma}_j \cdot \bar{\gamma}_j)$$

holds for all finite sets of constant curvatures and transverse shears, see Sections 5 and 6. Our bounding method follows the approach of Lipton (1994a) given in the context of reinforced Kirchoff plates. The approach taken here is distinct from the usual Hashin-Shtrikman method for obtaining bounds. Unlike the Hashin-Shtrikman method, we do not use a comparison material or solve an associated homogeneous elastic problem. Instead, we tackle the problem head on and write an integral equation for the local curvatures and transverse shears. The equations relate volume averaged curvatures and transverse shears to local curvatures and transverse shears through integral operators. The spectrum of these operators is analyzed, and it is shown that the operators are positive definite (see the Appendix). The positivity of the operators is used to obtain a Hilbert space variational principle for the effective stiffnesses. For the simple choice of constant trial fields, we arrive at the upper and lower estimates on the effective properties in terms of $\theta_2$ and $\mu$. We remark that the integral operators appearing here are the analogues of those introduced in the work of Golden and Papanicolaou (1983), for problems of heat conduction. In fact, the spectral estimates given in the appendix, allow one to write explicit analytic representation formulas for the effective rigidity. This topic is not pursued here, however we note that bounds obtained from such formulae would naturally agree with the ones presented here. Lastly we note that the usual Hashin-Shtrikman method can be used to obtain the bounds presented here.

To complete the characterization of the extremal set of rigidity tensors we introduce a special class of effective rigidities associated with hierarchical "laminar" arrangements of stiffeners, see Section 5. These are the analogies of the well known finite rank laminar micro geometries introduced in the contexts of elasticity and heat conductivity by Francfort and Murat (1986), Tartar (1985), and Lurie *et al.* (1982).

The effective tensors for such geometries have been found to be extremal in the context of Kirchoff plate theory and elasticity, see Gibianskii and Cherkaev (1984), Avellaneda (1987), Milton and Kohn (1988), Kohn and Lipton (1988), Allaire and Kohn (1993), and Lipton (1988). For every choice of volume fraction $\theta_2$ and probability measure $\mu$ we exhibit laminates whose effective rigidity tensors are identical to the tensors $(D_B^-, D_S^-)$ or $(D_B^+, D_S^+)$, see Sections 5 and 6. These observations provide us with the necessary closed form characterization of the set of extremal effective rigidities. This characterization has been used in the numerical approach to optimal reinforcement of Reisner-Mindlin plates given in the work of Díaz *et al.* (1995).

The paper is organized as follows: in Section 2 we introduce the effective rigidities in the context of periodic microstructure. To obtain bounds we write the effective tensors in terms of positive definite integral operators, see Section 3. Explicit formulae for the operators are obtained in Section 3. The positive definiteness of the operators is used to obtain a Hilbert space variational principle describing the effective rigidities, see Section 4. Bounds are obtained from the principles via suitable choices of trial fields. Closed form characterization of the sets of extremal rigidities are given in Sections 5 and 6.

## 2. EFFECTIVE RIGIDITY TENSORS

For our purposes we consider a unit period cell $\bar{Y}$ in $R^2$. Let $P_h$ and $P_s$ denote the projections onto the spaces of hydrostatic and shear strains, respectively. Then the rigidity tensors of the unreinforced and reinforced plate are given by $(D_B^1, D_S^1)$ and $(D_B^2, D_S^2)$, respectively, where:

$$D_B^i = (2/3)h_i^3 E((1+\nu)^{-1}P_s + (1-\nu)^{-1}P_h), \quad i = 1,2,$$

$$D_S^i = h_i E(1+\nu)^{-1} I, \quad i = 1,2,$$

and $I$ is the $2\times 2$ identity. To fix ideas we have assumed that the Young's modulus and Poisson ratio of the stiffener are identical to those of the plate. However, the methods given here extend to the case when the stiffener and plate have different isotropic elastic properties. The periodic rigidities are given by

$$(D_B, D_S) = \chi_1(D_B^1, D_S^1) + \chi_2(D_B^2, D_S^2) \tag{1}$$

where $\chi_2$ is the indicator function of the stiffeners and $\chi_1 = 1-\chi_2$. The volume fraction of stiffeners $\theta_2$ is given by

$$\theta_2 = \int_{\bar{Y}} \chi_2 \, \mathrm{d}Y.$$

We denote the average curvature and transverse shear by $\bar{\kappa}$ and $\bar{\gamma}$, respectively. The local curvature is given by $\kappa = \bar{\kappa} + \overset{*}{\kappa}$ where

$$\overset{*}{\kappa} = \bar{\nabla}\beta \equiv (1/2)(\partial_{y_j}\beta_i + \partial_{y_i}\beta_j),$$

and $\beta$ is the mean zero part of the transverse fiber rotation. The local transverse shear strain is $\gamma = \bar{\gamma} + \overset{*}{\gamma}$ where

$$\overset{*}{\gamma} = \nabla w - \beta.$$

Here $w$ is the transverse displacement of the mid-plane.

The effective properties in bending $D_B^e$, and shear $D_B^e$ are defined via equations

$$D_B^e \bar{\kappa} = \int_{\bar{Y}} D_B(\bar{\kappa} + \overset{*}{\kappa}) \, \mathrm{d}Y \tag{2}$$

$$D_S^e \bar{\gamma} = \int_{\bar{Y}} D_S(\bar{\gamma} + \overset{*}{\gamma}) \, \mathrm{d}Y. \tag{3}$$

Lastly, the fluctuating parts of the local fields are seen to satisfy the equilibrium equations

$$\int_Y D_B(\bar{\kappa}+\overset{*}{\kappa}):\hat{\kappa}\,\mathrm{d}Y+\int_Y D_S(\bar{\gamma}+\overset{*}{\gamma})\cdot\hat{\gamma}\,\mathrm{d}Y=0 \tag{4}$$

for all square integrable mean zero curvatures $\hat{\kappa}$ and shear strains $\hat{\gamma}$.

## 3. EFFECTIVE PROPERTIES IN TERMS OF INTEGRAL OPERATORS

In this section we provide a second formulation for the effective rigidity tensor in terms of integral operators. This will be used in the sequel to develop a suitable variational principle describing the effective stiffnesses. We introduce the operators $A_S$ and $A_B$ defined by

$$A_B\bar{\kappa}=(D_B^2-D_B^1)(\overset{*}{\kappa}+\bar{\kappa})$$

and

$$A_S\bar{\gamma}=(D_S^2-D_S^1)(\overset{*}{\gamma}+\bar{\gamma}).$$

From the cell problem (4) we have

$$\nabla\cdot(D_B(\overset{*}{\kappa}+\bar{\kappa}))=0$$

or, equivalently,

$$\nabla\cdot[(D_B^2+\chi_1(D_B^1-D_B^2))(\bar{\kappa}+\overset{*}{\kappa})]=0$$

since

$$D_B=\chi_1 D_B^1+\chi_2 D_B^2=D_B^2+\chi_1(D_B^1-D_B^2).$$

On the other hand, $\overset{*}{\kappa}=\bar{\nabla}\beta$ and it follows that

$$\nabla\cdot D_B^2\bar{\nabla}\beta=-\nabla\cdot[\chi_1(D_B^1-D_B^2)(\bar{\kappa}+\overset{*}{\kappa})]$$

or

$$\beta=-(\nabla\cdot D_B^2\bar{\nabla})^{-1}\nabla\cdot[\chi_1(D_B^1-D_B^2)(\bar{\kappa}+\overset{*}{\kappa})]$$

and

$$\overset{*}{\kappa}=\bar{\nabla}\beta=-\bar{\nabla}(\nabla\cdot D_B^2\bar{\nabla})^{-1}\nabla\cdot[\chi_1(D_B^1-D_B^2)(\bar{\kappa}+\overset{*}{\kappa})].$$

We denote the space of $2\times 2$, $Y$-periodic, square integrable symmetric matrix-valued fields by $H_B$. The space of $Y$-periodic, square integrable vector fields is denoted by $H_S$. For any $q$ in $H_B$ we define $P_B$ as the operator

$$P_Bq=\bar{\nabla}(\nabla\cdot D_B^2\bar{\nabla})^{-1}\nabla\cdot q$$

so that

$$\overset{*}{\kappa}+\bar{\kappa}=\bar{\kappa}-P_B\chi_1(D_B^1-D_B^2)(\bar{\kappa}+\overset{*}{\kappa})$$

and formally, we have

$$\bar{\kappa}+\overset{*}{\kappa} = [\mathbf{I}-P_B\chi_1(D_B^2-D_B^1)]^{-1}(\bar{\kappa}).$$

Since

$$(D_B^2-D_B^1)(\overset{*}{\kappa}+\bar{\kappa}) = A_B\bar{\kappa}$$

we obtain the final expression

$$A_B = (D_B^2-D_B^1)[\mathbf{I}-P_B\chi_1(D_B^2-D_B^1)]^{-1}. \tag{5}$$

We proceed in identical fashion to derive

$$A_S = (D_S^2-D_S^1)[I-P_S\chi_1(D_S^2-D_S^1)]^{-1} \tag{6}$$

where the operator $P_S$ is defined on the space of $Y$-periodic square integrable vector fields and is given by

$$P_S p = \nabla(\nabla\cdot D_S^2\nabla)^{-1}\nabla\cdot p$$

Applying (1), (2) and (3) write

$$(D_B^2-D_B^e)\bar{\kappa}:\bar{\kappa} = \int_Y \chi_1(D_B^2-D_B^1)(\bar{\kappa}+\overset{*}{\kappa})\,\mathrm{d}Y$$

$$(D_S^2-D_S^e)\bar{\gamma}\cdot\bar{\gamma} = \int_Y \chi_1(D_S^2-D_S^1)(\bar{\gamma}+\overset{*}{\gamma})\,\mathrm{d}Y. \tag{7}$$

From the definitions of $A_B$ and $A_S$, it follows that

$$(D_B^2-D_B^e)\bar{\kappa}:\bar{\kappa} = \int_Y \chi_1 A_B\bar{\kappa}:\bar{\kappa}\,\mathrm{d}Y \tag{8}$$

and

$$(D_S^2-D_S^e)\bar{\gamma}\cdot\bar{\gamma} = \int_Y \chi_1 A_S\bar{\gamma}\cdot\bar{\gamma}\,\mathrm{d}Y. \tag{9}$$

The identities (8) and (9) provide a second formulation for the effective rigidity tensor. The operators $A_B$ and $A_S$ are shown in the Appendix to be well defined and positive definite.

## 4. EXPLICIT FORMULAE FOR NONLOCAL OPERATORS

Here we find explicit formulas for the nonlocal operators $P_S$ and $P_B$ using Fourier methods. Consider first the equation

$$(\nabla\cdot D_S^2\nabla)w = \nabla\cdot v \tag{10}$$

where $v$ is $Y$-periodic with mean zero. Write $\nabla\cdot v$ as

$$\nabla\cdot v = \sum_{k\neq 0} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{y}} k\cdot\hat{v}(k)$$

and $w$ as

$$w = \sum_{k \neq 0} e^{i\mathbf{k}\cdot\mathbf{y}} \hat{w}(k) \tag{11}$$

where

$$\hat{w}(k) = -\frac{ik \cdot \hat{v}(k)}{k \cdot (D_S^2 k)}. \tag{12}$$

Recalling the definition of the operation $P_S$, it follows from (10), (11) and (12) that:

$$P_S v = \nabla w = \sum_{k \neq 0} i e^{i\mathbf{k}\cdot\mathbf{y}} \hat{w}(k) k = \sum_{k \neq 0} \frac{k \cdot \hat{v}(k)}{k \cdot (D_S^2 k)} e^{i\mathbf{k}\cdot\mathbf{y}} k.$$

Rearrangement gives,

$$P_S v = \sum_{k \neq 0} e^{i\mathbf{k}\cdot\mathbf{y}} \hat{P}_S(k) \hat{v}(k)$$

where

$$\hat{P}_S(k) = \frac{k \otimes k}{k \cdot (D_S^2 k)}.$$

Proceeding in a similar way we obtain

$$P_B p = \sum_{k \neq 0} e^{i\mathbf{k}\cdot\mathbf{y}} \hat{P}_B(\hat{k}) \hat{p}(k)$$

where

$$\hat{k} = \frac{k}{|k|}, \quad c_0 = \frac{3(1+\nu)}{2h_2^3 E},$$

and

$$\hat{P}_B(\hat{k}) = c_0[\tfrac{1}{2}(\delta_{mo}\hat{k}_n\hat{k}_p + \delta_{mp}\hat{k}_n\hat{k}_o + \delta_{no}\hat{k}_m\hat{k}_p + \delta_{np}\hat{k}_m\hat{k}_o) - (1+\nu)(\hat{k}_m\hat{k}_n\hat{k}_o\hat{k}_p)].$$

## 5. UPPER BOUNDS ON THE EFFECTIVE RIGIDITY

In the section we introduce a Hilbert Space variational principle describing the effective rigidity. This principle is used to obtain upper bounds on the effective properties. From (8) and (9) it follows that for any constant curvature $\bar{\kappa}$ and transverse shear $\bar{\gamma}$ that

$$(D_B^2 - D_B^e)\bar{\kappa} : \bar{\kappa} + (D_S^2 - D_S^e)\bar{\gamma} \cdot \bar{\gamma} = \int_{Y} \chi_1 \begin{bmatrix} A_B & 0 \\ 0 & A_S \end{bmatrix} \mathrm{d}Y \begin{pmatrix} \bar{\kappa} \\ \bar{\gamma} \end{pmatrix} \cdot \begin{pmatrix} \bar{\kappa} \\ \bar{\gamma} \end{pmatrix}.$$

We observe that the operators $A_B$ and $A_S$ are positive definite (see Appendix), from which it follows

$$\int_{Y} \chi_1 \begin{bmatrix} A_B & 0 \\ 0 & A_S \end{bmatrix} \begin{pmatrix} \bar{\kappa} - A_B^{-1} p \\ \bar{\gamma} - A_S^{-1} q \end{pmatrix} \cdot \begin{pmatrix} \bar{\kappa} - A_B^{-1} p \\ \bar{\gamma} - A_S^{-1} q \end{pmatrix} \mathrm{d}T \geq 0. \tag{13}$$

Expanding (13) gives the variational principle

$$(D_B^2-D_B^e)\bar{\kappa}:\bar{\kappa}+(D_S^2-D_S^e)\bar{\gamma}\cdot\bar{\gamma}\geqslant 2\int_Y \chi_1\begin{pmatrix}p\\q\end{pmatrix}\cdot\begin{pmatrix}\bar{\kappa}\\\bar{\gamma}\end{pmatrix}\mathrm{d}Y-\int_Y \chi_1\begin{bmatrix}A_B^{-1} & 0\\ 0 & A_S^{-1}\end{bmatrix}\begin{pmatrix}p\\q\end{pmatrix}\cdot\begin{pmatrix}p\\q\end{pmatrix}\mathrm{d}Y$$

for any $q$ in $H_B$ and $p$ in $H_S$. It is easily seen that this variational principle is tight. Indeed, one has equality for the choice $p = A_B\bar{\kappa}$ and $q = A_S\bar{\gamma}$. The variational principle is used to obtain lower bounds. We consider a constant $2\times 2$ matrix $\eta$ and constant vector $\xi$ in $R^2$. We let $q = \eta$ and $p = \xi$ and apply (5) and (6) to obtain

$$(D_B^2-D_B^e)\bar{\kappa}:\bar{\kappa}+(D_S^2-D_S^e)\bar{\gamma}\cdot\bar{\gamma}\geqslant 2\theta_1(\eta:\bar{\kappa}+\xi\cdot\bar{\gamma})-\int_Y \chi_1(D_B^2-D_B^1)^{-1}\eta:\eta\,\mathrm{d}Y$$
$$+\int_Y \chi_1 P_B\chi_1\eta:\eta\,\mathrm{d}Y-\int_Y \chi_1(D_S^2-D_S^1)^{-1}\,\mathrm{d}Y\xi\cdot\xi+\int_Y \chi_1 P_S\chi_1\xi\cdot\xi\,\mathrm{d}Y.$$

Expanding the tensors

$$\int_Y \chi_1 P_B\chi_1\,\mathrm{d}Y \quad\text{and}\quad \int_Y \chi_1 P_S\chi_1\,\mathrm{d}Y,$$

gives

$$\int_Y \chi_1 P_B\chi_1\,\mathrm{d}Y=\sum_{k\neq 0}|\hat{\chi}_1(k)|^2\hat{P}_B(k)$$

and

$$\int_Y \chi_1 P_S\chi_1\,\mathrm{d}Y=\sum_{k\neq 0}|\hat{\chi}_1(k)|^2\hat{P}_S(k)$$

where

$$\sum_{k\neq 0}|\hat{\chi}_1(k)|^2=\theta_1\theta_2. \tag{14}$$

We find now that

$$(D_B^2-D_B^e)\bar{\kappa}:\bar{\kappa}+(D_S^2-D_S^e)\bar{\gamma}\cdot\bar{\gamma}\geqslant 2\theta_1(\eta:\kappa+\xi\cdot\gamma)-[\theta_1(D_B^2-D_B^1)^{-1}$$
$$-\sum_{k\neq 0}|\hat{\chi}_1(k)|^2\hat{P}_B(k)]\eta:\eta-[\theta_1(D_S^2-D_S^1)^{-1}-\sum_{k\neq 0}|\hat{\chi}_1(k)|^2\hat{P}_S(k)]\xi\cdot\xi \tag{15}$$

which holds for any $\xi$ and $\eta$. Introduce now tensors $D_S^+$ and $D_S^+$ defined via

$$\theta_1(D_B^2-D_B^+)^{-1}=(D_B^2-D_B^1)^{-1}-\theta_2\sum_{k\neq 0}\frac{1}{\theta_1\theta_2}|\hat{\chi}_1(k)|^2\hat{P}_B(\hat{k}) \tag{16}$$

and

$$\theta_1(D_S^2 - D_S^+)^{-1} = (D_S^2 - D_S^1)^{-1} - \theta_2 \sum_{k \neq 0} \frac{1}{\theta_1 \theta_2} |\hat{\chi}_1(k)|^2 \hat{P}_S(\hat{k}). \tag{17}$$

Using $D_S^+$ and $D_S^+$ we may write (15) as

$$(D_B^2 - D_B^e)\bar{\kappa} : \bar{\kappa} + (D_S^2 - D_S^e)\bar{\gamma} \cdot \bar{\gamma} \geqslant 2\theta_1(\eta : \kappa + \xi \cdot \gamma) - \theta_1^2(D_B^2 - D_B^+)\eta : \eta - \theta_1^2(D_S^2 - D_S^+)\xi \cdot \xi \quad \forall \xi, \eta \tag{18}$$

and hence,

$$(D_B^2 - D_B^e)\bar{\kappa} : \bar{\kappa} + (D_S^2 - D_S^e)\bar{\gamma} \cdot \bar{\gamma} \geqslant \sup_{\xi,\eta} [2\theta_1(\eta : \kappa + \xi \cdot \gamma) - \theta_1^2(D_B^2 - D_B^+)\eta : \eta - \theta_1^2(D_S^2 - D_S^+)\xi \cdot \xi]. \tag{19}$$

From stationarity, the supremum (19) is achieved when

$$\eta = \frac{1}{\theta_1}(D_B^2 - D_B^+)\bar{\kappa}$$

and

$$\xi = \frac{1}{\theta_1}(D_S^2 - \bar{D}_S^+)\bar{\gamma}.$$

From this choice the bound is given by

$$(D_B^2 - D_B^e)\bar{\kappa} : \bar{\kappa} + (D_S^2 - D_S^e)\bar{\gamma} \cdot \bar{\gamma} \leqslant (D_B^2 - D_B^+)\bar{\kappa} : \bar{\kappa} + (D_S^2 - D_S^+)\bar{\gamma} \cdot \bar{\gamma}.$$

In the next section we show that the tensors $D_S^+$ and $D_S^+$ in (16) and (17) correspond to the effective properties of a suitably chosen finite rank laminate. To facilitate this identification, we follow Willis (1982) and Avellaneda and Milton (1989), and write the sums in (16) and (17) as

$$\sum_{k \neq 0} \frac{1}{\theta_1 \theta_2} |\hat{\chi}_1(k)|^2 \hat{P}_B(\hat{k}) = \int_{S^1} \hat{P}_B(n)\, \mathrm{d}\mu(n)$$

$$\sum_{k \neq 0} \frac{1}{\theta_1 \theta_2} |\hat{\chi}_1(k)|^2 \hat{P}_S(\hat{k}) = \int_{S^1} \hat{P}_S(n)\, \mathrm{d}\mu(n)$$

where $n$ is a unit vector on the circle $S^1$ and the positive correlation measure $\mu$ is given by

$$\mathrm{d}\mu(n) = \sum_{|l|=1} \frac{1}{\theta_1 \theta_2} \sum_{\|k\|=1} |\hat{\chi}_1(k)|^2 \delta(l-n)\, \mathrm{d}n,$$

and from (14) it follows that $\int_{S^1} \mathrm{d}\mu(n) = 1$.

We indicate the dependence of the tensors $D_B^+$ and $D_S^+$ on area fraction of the stiffener reinforced plate $\theta_2$ and the correlation measure $\mu$ by writing, $D_B^+(\mu, \theta_2)$, $D_S^+(\mu, \theta_2)$ and (16),

(17) are written

$$D_B^+(\mu,\theta_2) = D_B^2 - \left(\theta_1^{-1}(D_B^2 - D_B^1)^{-1} - \frac{\theta_2}{\theta_1}\int_{S^1} \hat{P}_B(n)\,\mathrm{d}\mu(n)\right)^{-1} \tag{20}$$

$$D_S^+(\mu,\theta_2) = D_S^2 - \left(\theta_1^{-1}(D_S^2 - D_S^1)^{-1} - \frac{\theta_2}{\theta_1}\int_{S^1} \hat{P}_S(n)\,\mathrm{d}\mu(n)\right)^{-1}. \tag{21}$$

Collecting results it follows from (5) that for composites with specified volume fraction $\theta_2$ and correlation measure $\mu$ we arrive at the:

Theorem 5.1. For given values of volume fraction and anisotropy measure one has:

$$D_B^e \bar{\kappa} : \bar{\kappa} + D_S^e \bar{\gamma} \cdot \bar{\gamma} \leqslant \bar{D}_B^+ \bar{\kappa} : \bar{\kappa} + \bar{D}_S^+ \bar{\gamma} \cdot \bar{\gamma}$$

for all $2 \times 2$ symmetric matrices $\bar{\kappa}$ and vectors $\bar{\gamma}$ in $R^2$.

## 6. CHARACTERIZATION OF EFFECTIVE TENSORS MAXIMIZING SUMS OF ELASTIC ENERGIES

To complete the characterization of the set of external effective rigidity tensors we introduce a special class of rib geometries whose effective rigidities will prove to be extremal.

We now introduce the notion of a finite rank stiffener reinforced Mindlin plate. To fix ideas we describe a rank 2 reinforced plate. We consider a family of uniformly spaced ribs of thickness $2h_2$ normal to a given direction $n_1$. The family is assumed to oscillate on a scale of order $\varepsilon^2$. Next we consider strips of order $\varepsilon$ containing the finitely ribbed material. The normal to the strips is specified by $n_2$.

These strips are uniformly interleaved with the stiffened plate on a scale of order $\varepsilon$. The effective properties are obtained asymptotically in the $\varepsilon = 0$ limit. Higher rank stiffeners are defined iteratively. We provide explicit formulas for $\bar{D}_B$ and $\bar{D}_S$ for such micro structures.

One observes that the formulae (20) and (21) are mathematically analogous to those defining effective heat conductivity and elasticity. Indeed, a direct transcription of the finite rank laminate formulae of Murat and Tartar (1985) for heat conductivity and those of Francfort and Murat (1986) for elasticity deliver the following formulas for $\bar{D}_B$ and $\bar{D}_S$

$$\bar{D}_B(\nu,\theta_2) = D_B^2 - \left(\theta_1^{-1}(D_B^2 - D_B^1)^{-1} - \frac{\theta_2}{\theta_1}\int_{S^1} \hat{P}_B(n)\,\mathrm{d}\nu(n)\right)^{-1} \tag{22}$$

$$\bar{D}_S(\nu,\theta_2) = D_S^2 - \left(\theta_1^{-1}(D_S^2 - D_S^1)^{-1} - \frac{\theta_2}{\theta_1}\int_{S^1} \hat{P}_S(n)\,\mathrm{d}\nu(n)\right)^{-1} \tag{23}$$

where the positive measure $\nu(n)$ on the unit circle $S^1$ is defined by

$$\nu(n) = \sum_{i=1}^{J} \rho_i \delta(n - n_i).$$

The extremal nature of the effective rigidity tensors of finite rank stiffener reinforced plates is seen in the following:

Lemma 6.1. For prescribed $\theta_2$ and correlation measure $\mu$, the geometric tensors $D_B^+(\mu,\theta_2)$, $D_S^+(\mu,\theta_2)$ correspond to the effective bending stiffness and transverse shear stiffness of a suitably constructed finite rank stiffener reinforced plate.

To prove the Lemma we follow the approach given by Avellaneda (1987) in the context of two phase elasticity. Let $p$ be any correlation measure and we consider the set **S** of all pairs of the form

$$\left(\int_{S^1} \hat{P}_B(n)\,\mathrm{d}p(n),\quad \int_{S^1} \hat{P}_S(n)\,\mathrm{d}p(n)\right).$$

We introduce the 6-dimensional space $L$ of totally symmetric fourth order tensors and the 3-dimensional space $T$ of symmetric $2\times 2$ matrices. The surface $S$ is defined by the map

$$(\hat{P}_B(n), \hat{P}_S(n)) : S^1 \to L \times T$$

and we consider the convex hull of the surface denoted by $C(S)$. It is evident that **S** is a subset of convex hull $C(S)$. From the definition of $C(S)$ we have that all extreme points lie on the surface $S$ contained in the 9-dimensional space $L \times T$. Thus, for a given correlation measure $\mu$ it follows from Carathedory's theorem that there exists a laminate with measure $v$ of at most rank 10 for which

$$\left(\int_{S^1} \hat{P}_B(n)\,\mathrm{d}v(n),\quad \int_{S^1} \hat{P}_S(n)\,\mathrm{d}v(n)\right) = \left(\int_{S^1} \hat{P}_B(n)\,\mathrm{d}\mu(n),\quad \int_{S^1} \hat{P}_S(n)\,\mathrm{d}\mu(n)\right). \tag{24}$$

The Lemma follows immediately from (24) and the formulae for $D_B^+$, $D_S^+$ and $\bar{D}_B$, $\bar{D}_S$.

Combining Theorem 5.1 and lemma 6.1 gives:

Theorem 6.1. For all composites with prescribed volume fraction of stiffeners $\theta_2$ and anisotropy measure $\mu$, there exists an effective rigidity tensor $(\bar{D}_B, \bar{D}_S)$ of a finite rank stiffener reinforced plate for which

$$\sum_{i=1}^{N} (D_B^e \bar{\kappa}_i : \bar{\kappa}_i + D_S^e \bar{\gamma}_i \cdot \bar{\gamma}_i) \leqslant \sum_{i=1}^{N} (\bar{D}_B \bar{\kappa}_i : \bar{\kappa}_i + \bar{D}_S \bar{\gamma}_i \cdot \bar{\gamma}_i)$$

holds, for any set of constant curvatures $\bar{\kappa}_1, \bar{\kappa}_2, \ldots, \bar{\kappa}_N$ and transverse shears $\bar{\gamma}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_N$.

This theorem shows that extremal rigidities maximizing sums of energies can be found within the class of finite rank laminates.

Theorem 6.2. For fixed area fraction of stiffeners $\theta_2$ and for a given set of curvatures $\bar{\kappa}_1, \bar{\kappa}_2, \ldots, \bar{\kappa}_N$ and transverse shears $\bar{\gamma}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_N$ one has the upper bound:

$$\sum_{i=1}^{N} (D_B^e \bar{\kappa}_i : \bar{\kappa}_i + D_S^e \bar{\gamma}_i) \leqslant \max_{v}\left(\sum_{i=1}^{N} (\bar{D}_B \bar{\kappa}_i : \bar{\kappa}_i + \bar{D}_S \bar{\gamma}_i \cdot \bar{\gamma}_i)\right).$$

## 7. NEW VARIATIONAL PRINCIPLES AND ENERGY MINIMIZING SETS OF EFFECTIVE RIGIDITIES

We develop lower variational principles for the effective rigidity tensor. From these we follow the procedure given in Sections 3–5 to describe the set of energy minimizing effective rigidities. Introducing the operators $C_S$ and $C_B$ defined by

$$C_B \bar{\kappa} = (D_B^2 - D_B^1)[I + P_B \chi_2 (D_B^2 - D_B^1)]^{-1}$$

and

$$C_S\bar{\gamma} = (D_S^2 - D_S^1)[I + P_S\chi_2(D_S^2 - D_S^1)]^{-1}.$$

The effective rigidity tensor is written

$$(D_B^e - D_B^1)\bar{\kappa} : \bar{\kappa} + (D_S^e - D_S^1)\bar{\gamma} \cdot \bar{\gamma} = \int_Y \chi_2 \begin{bmatrix} C_B & 0 \\ 0 & C_S \end{bmatrix} \begin{pmatrix} \bar{\kappa} \\ \bar{\gamma} \end{pmatrix} \cdot \begin{pmatrix} \bar{\kappa} \\ \bar{\gamma} \end{pmatrix} \mathrm{d}Y. \tag{25}$$

One can show as before that the transforms $C_B$ and $C_S$ are positive definite. Proceeding as in Section 4 we obtain the variational principle:

$$(D_B^e - D_B^1)\bar{\kappa} : \bar{\kappa} + (D_S^e - D_S^1)\bar{\gamma} \cdot \bar{\gamma} \geqslant 2 \int_Y \chi_2 \begin{pmatrix} q \\ p \end{pmatrix} \cdot \begin{pmatrix} \bar{\kappa} \\ \bar{\gamma} \end{pmatrix} \mathrm{d}Y - \int_Y \chi_2 \begin{bmatrix} C_B^{-1} & 0 \\ 0 & C_S^{-1} \end{bmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \cdot \begin{pmatrix} q \\ p \end{pmatrix} \mathrm{d}Y \tag{26}$$

for all $q$ in $H_B$ and $p$ in $H_S$. One can use hierarchical grooved geometries to construct compliant effective rigidities. Here we start with a thick plate (of thickness $h_2$) and cut uniformly spaced grooves of depth $h_2 - h_1$ normal to a given direction $n_1$. The grooves are assumed to oscillate on a scale of order $\varepsilon^N$. Next we consider strips of order $\varepsilon^{N-1}$ containing the grooved material. The normal to the strips is denoted by $n_2$. These strips are uniformly interleaved with wider grooves on a scale of order $\varepsilon^{N-1}$. This process is carried out iteratively until we arrive at a structure with characteristic length $\varepsilon$. The effective rigidity is obtained in the $\varepsilon = 0$ limit. The formulae for the effective rigidities $(\underline{D}_B, \underline{D}_S)$ are given by

$$\underline{D}_B = D_B^1 + \left(\theta_2^{-1}(D_B^2 - D_B^1)^{-1} + \frac{\theta_1}{\theta_2} \int_{S^1} \hat{P}_B(n)\,\mathrm{d}\nu(n)\right)^{-1}$$

$$\underline{D}_S = D_S^1 + \left(\theta_2^{-1}(D_S^2 - D_S^1)^{-1} + \frac{\theta_1}{\theta_2} \int_{S^1} \hat{P}_S(n)\,\mathrm{d}\nu(n)\right)^{-1}.$$

Proceeding as in the previous sections we obtain the following theorems.

Theorem 7.1. For all composites with prescribed volume fraction of grooves $\theta_1$ and anisotropy measure $\mu$, there exists an effective rigidity tensor $(D_B, D_S)$ associated with a finite rank grooved plate for which

$$\sum_{i=1}^{N} (\underline{D}_B\bar{\kappa}_i : \bar{\kappa}_i + \bar{D}_S\bar{\gamma}_i \cdot \bar{\gamma}_i) \leqslant \sum_{i=1}^{N} (D_B^e\bar{\kappa}_i : \bar{\kappa}_i + D_S^e\bar{\gamma}_i \cdot \bar{\gamma}_i)$$

holds, for any set of constant curvatures $\bar{\kappa}_1, \bar{\kappa}_2, \ldots, \bar{\kappa}_N$ and transverse shears $\bar{\gamma}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_N$.

This theorem shows that extremal rigidities minimizing sums of energies can be found within the class of finite rank laminates.

Theorem 7.2. For fixed area fraction of grooves $\theta_1$ and for a given set of curvatures $\bar{\kappa}_1, \bar{\kappa}_2, \ldots, \bar{\kappa}_N$ and transverse shears $\bar{\gamma}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_N$ one has the lower bound:

$$\min_{\nu}\left(\sum_{i=1}^{N} (\bar{D}_B\bar{\kappa}_i : \bar{\kappa}_I + \bar{D}_S\bar{\gamma}_i \cdot \bar{\gamma}_i)\right) \leqslant \sum_{i=1}^{N} (D_B^e\bar{\kappa}_i : \bar{\kappa}_i + D_S^e\bar{\gamma}_i \cdot \bar{\gamma}_i).$$

## 8. CONCLUDING REMARKS

The effective stiffnesses of finite rank reinforced plates can be written in terms of four independent scalar variables, these being moments of trigonometric functions, see Díaz *et al.* (1995). Such transformations have been introduced earlier and the set of moments characterized for problems in two dimensional elasticity and Kirchoff plate theory by Avellaneda and Milton (1989). The inverse problem of finding layer widths and orientations from the moments is solved in Lipton (1994b). These results can be applied in the present context to show that at most third rank stiffener reinforced plates span the extremal set of effective tensors. This was done in the recent work of Díaz *et al.* (1995).

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## APPENDIX

We establish invertibility and positivity for the operators $A_B$ and $A_S$ introduced in Section 3. We start by formally writing $A_B$ and $A_S$ as:

$$A_B = \sqrt{\delta_B} L_B^{-1} \sqrt{\delta_B}, \quad A_S = \sqrt{\delta_S} L_S^{-1} \sqrt{\delta_S},$$

where

$$\delta_B = D_B^2 - D_B^1, \quad \delta_S = D_S^2 - D_S^1$$

and

$$L_B = I - \sqrt{\delta_B} P_B \chi_1 \sqrt{\delta_B}, \quad L_S = I - \sqrt{\delta_S} P_S \chi_1 \sqrt{\delta_S}.$$

Here $\delta_B$ and $\delta_S$ are positive definite and so their square roots are well defined. We establish that $A_B$ and $A_S$ are well defined and invertible by showing that the inverses $L_B^{-1}$ and $L_S^{-1}$ exist. We denote the usual inner products for the spaces $H_B$ and $H_S$ by $\langle \cdot, \cdot \rangle_B$ and $\langle \cdot, \cdot \rangle_S$, respectively. One easily sees that the operators $L_B$ and $L_S$ are symmetric and invertibility for the linear operators $L_B$ and $L_S$ follows from the spectral estimates:

$$\langle L_B q, q \rangle_B \geqslant (1 - t_B) \langle q, q \rangle_B, \tag{A.1}$$

$$\langle L_B q, L_B q \rangle_B \leqslant 2(1 + t_B^2) \langle q, q \rangle_B, \tag{A.2}$$

for all $q$ in $H_B$ and

$$\langle L_S p, p \rangle_S \geqslant (1 - t_S) \langle p, p \rangle_S \tag{A.3}$$

$$\langle L_S p, L_S p \rangle_S \leqslant 2(1 + t_S^2) \langle p, p \rangle_S \tag{A.4}$$

for all $p$ in $H_S$.

Here $t_B$ and $t_S$ are positive, satisfy $t_B \leqslant 1$, $t_S \leqslant 1$ and are given by

$$t_B = \frac{h_2^3 - h_1^3}{h_2^3}, \quad t_S = \frac{h_2 - h_1}{h_2}. \tag{A.5}$$

To fix ideas we show how to obtain the estimates (A.1) and (A.2) on $L_B$. Expansion of $\langle L_B q, q \rangle_B$, noting that $\langle \chi_1 q, q \rangle_B \leqslant \langle q, q \rangle_B$ and application of Cauchy's inequality gives:

$$\langle L_B q, q \rangle_B \geqslant \langle q, q \rangle_B - \langle q, q \rangle_B^{1/2} \langle (\sqrt{\delta_B} P_B \sqrt{\delta_B})^2 q, q \rangle_B^{1/2}. \tag{A.6}$$

Expansion of $\langle L_B q, L_B q \rangle_B$ and application of Cauchy's inequality gives

$$\langle L_B q, L_B q \rangle_B \leqslant 2\{\langle q, q \rangle + \langle (\sqrt{\delta_B} P_B \sqrt{\delta_B})^2 q, q \rangle_B\}. \tag{A.7}$$

From (A.6) and (A.7) we see that (A.1) and (A.2) follow easily from the following estimate:

$$\langle (\sqrt{\delta_B} P_B \sqrt{\delta_B})^2 q, q \rangle_B \leqslant t_B^2 \langle q, q \rangle_B. \tag{A.8}$$

We remark that (A.8) amounts to an upper bound on the eigenvalues for the operator $\sqrt{\delta_B} P_B \sqrt{\delta_B}$. To obtain (A.8) we apply Parseval's identity to write

$$\langle (\sqrt{\delta_B} P_B \sqrt{\delta_B}) q, q \rangle_B = \sum_{k \neq 0} (\sqrt{\delta_B} \hat{P}_B(k) \sqrt{\delta_B}) \hat{q}(k) : \overline{\hat{q}(k)}. \tag{A.9}$$

We estimate each term in the series to find

$$(\sqrt{\delta_B} \hat{P}_B(k) \sqrt{\delta_B}) \hat{q}(k) : \overline{\hat{q}(k)} \leqslant t_B |\hat{q}(k)|^2, \tag{A.10}$$

and (A.8) follows from a second application of Parseval's identity.

The estimate (A.10) follows by computing the eigenvalues of the tensor $\sqrt{\delta_B} \hat{P}_B(k) \sqrt{\delta_B}$. The eigenvalues are found to be independent of the wave vector and have the values 0 and $t_B$. Here $t_B$ is an eigenvalue of multiplicity two. A computation shows that the operator $\sqrt{\delta_B} \hat{P}_B(k) \sqrt{\delta_B}$ is written:

$$\sqrt{\delta_B} \hat{P}(k) \sqrt{\delta_B} = t_B P(\hat{k}), \tag{A.11}$$

where $\hat{k} = k/|k|$ and $P(\hat{k})$ is the projection onto the subspace spanned by the matrices

$$\rho \hat{k}^\perp \otimes \hat{k}^\perp + \beta \hat{k} \otimes \hat{k}, \quad \frac{\sqrt{2}}{2} (\hat{k}^\perp \otimes \hat{k} + \hat{k} \otimes \hat{k}^\perp), \tag{A.12}$$

where

$$\rho = 1/2(\sqrt{1+v}-\sqrt{1-v}) \tag{A.13}$$

and

$$\beta = 1/2(\sqrt{1+v}+\sqrt{1-v}). \tag{A.14}$$

Last, we recover the necessary positivity properties for the transforms $A_B^{-1}$ and $A_S^{-1}$. These are:

$$\langle \chi_1 A_B^{-1} q, q\rangle_B \geqslant 0 \tag{A.15}$$

and

$$\langle \chi_1 A_S^{-1} p, p\rangle_S \geqslant 0 \tag{A.16}$$

We prove (A.15) noting (A.16) follows along the same lines. Expanding $A_B^{-1}$, we see that (A.15) is established through the following string of equalities:

$$\langle \chi_1 A_B^{-1} q, q\rangle_B = \langle \chi_1 (\sqrt{\delta_B})^{-1} L_B (\sqrt{\delta_B})^{-1} q, q\rangle_B \tag{A.17}$$

$$= \langle \chi_1 (\sqrt{\delta_B})^{-1} (I - \sqrt{\delta_B} P_B \chi_1 \sqrt{\delta_B})(\sqrt{\delta_B})^{-1} q, q\rangle_B \tag{A.18}$$

$$= \langle (I - \sqrt{\delta_B} P_B \chi_1 \sqrt{\delta_B})(\sqrt{\delta_B})^{-1} \chi_1 q, (\sqrt{\delta_B})^{-1} \chi_1 q\rangle_B \tag{A.19}$$

$$= \langle L_B (\sqrt{\delta_B})^{-1} \chi_1 q, (\sqrt{\delta_B})^{-1} \chi_1 q\rangle_B \tag{A.20}$$

$$\geqslant (1-t_B)\langle (\sqrt{\delta_B})^{-1} \chi_1 q, (\sqrt{\delta_B})^{-1} \chi_1 q\rangle_B, \tag{A.21}$$

where the last inequality follows from (A.1). It is evident that the necessary positivity properties for $A_B$ and $A_S$ follow immediately from those on $A_B^{-1}$ and $A_S^{-1}$.