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Optimal lower bounds on the dilatational strain inside random two-phase composites subjected to hydrostatic loading.

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Abstract. Composites made from two linear isotropic elastic materials are considered. It is assumed that only the volume fraction of each elastic material is known. The composite is subjected to a uniform hydrostatic strain. For this case lower bounds on all r^{th} moments of the dilatational strain field inside each phase are obtained for $r \geq 2$. A lower bound on the maximum value of the dilatational strain field is also obtained. These bounds are given in terms of the volume fractions of the component materials. All of these bounds are shown to be the best possible as they are attained by the dilatational strain field inside the Hashin–Shtrikman coated sphere assemblage. The bounds provide a new opportunity for the assessment of the local dilatational strain in terms of a statistical description of the microstructure.

Keywords. Composites, failure criteria, strain, random media.

1 Introduction

Failure initiation in composite materials is a multi-scale phenomena. Central to the analysis is the assessment of the local stress and strain fields generated by macroscopic forces. Quantities sensitive to local field behavior include higher order moments of the stress and strain fields inside the composite. These quantities have seen extensive application in the theoretical analysis of material failure, see Kelly and Macmillan (1986). Failure criteria are often associated with the deviatoric part of the elastic strain tensor. However critical dilatational deformation can preceed critical deviatoric deformation in polymers, see Asp, Berglund and Talerja (1996). The dilatational strain has recently been incorporated into failure criteria for epoxy matrix composites seen in aircraft, see Gosse and Christensen (2001).

The dilatational strain measures the local volumetric change associated with the local strain field and is given by

$$tr\{\epsilon(\mathbf{x})\}/3 = (\epsilon_{11}(\mathbf{x}) + \epsilon_{22}(\mathbf{x}) + \epsilon_{33}(\mathbf{x}))/3. \tag{1.1}$$

For planar elastic problems the dilatational strain reduces to $tr\{\epsilon(\mathbf{x})\}/2 = (\epsilon_{11}(\mathbf{x}) + \epsilon_{22}(\mathbf{x}))/2$. Both two dimensional and fully three dimensional problems are treated here and the associated dilatational strain is written as

$$tr\{\epsilon\}/d \tag{1.2}$$

where $d = 2$ or $d = 3$ corresponds to two dimensional or three dimensional elasticity respectively.

Here we consider a cube Q filled with two linearly elastic materials. Only the volume fractions and the elastic properties of the two phases are known. No other assumptions on the configuration of the two elastic materials inside the cube are made. The subdomains occupied by materials one and two are denoted by Q_1 and Q_2 respectively. The indicator function of material one is denoted

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by χ_1 and takes the value one inside Q_1 and zero outside. The indicator function of material two is given by χ_2 and $\chi_2 = 1 - \chi_1$. A constant hydrostatic strain field is applied to the cube and we consider the moments of the dilatational part of the strain field inside each phase given by

$$< \chi_1 |tr\{\epsilon(\mathbf{x})\}/d|^r >^{1/r} \quad \text{and} \quad < \chi_2 |tr\{\epsilon(\mathbf{x})\}/d|^r >^{1/r} \tag{1.3}$$

for $2 \leq r < \infty$ and $d = 2, 3$. Here angle brackets $< \cdot >$ denote the volume average over the cube. We also consider the L^∞ norms given by

$$\begin{aligned} \||tr\{\epsilon(\mathbf{x})\}/d|\|_{L^\infty(Q_1)} &= \lim_{r\to\infty} < \chi_1 |tr\{\epsilon(\mathbf{x})\}/d|^r >^{1/r} \\ \||tr\{\epsilon(\mathbf{x})\}/d|\|_{L^\infty(Q_2)} &= \lim_{r\to\infty} < \chi_2 |tr\{\epsilon(\mathbf{x})\}/d|^r >^{1/r} \\ \||tr\{\epsilon(\mathbf{x})\}/d|\|_{L^\infty(Q)} &= \lim_{r\to\infty} < |tr\{\epsilon(\mathbf{x})\}/d|^r >^{1/r} . \end{aligned} \tag{1.4}$$

In this paper optimal lower bounds on the moments of the dilatational strain are found when the composite is subjected to applied strains of the hydrostatic type $\bar{\epsilon} = \epsilon_0 I$, where I is the identity and ϵ_0 is a constant. In Section 4 we present explicit optimal lower bounds on the moments (1.3) and L^∞ norms (1.4) that are given in terms of the volume fractions, elastic properties of the materials and the imposed strain ϵ_0. It is shown that the minimizing configurations are given by Hashin and Shtrikman (1962) coated sphere or cylinder assemblages depending upon whether $d = 3$ or $d = 2$ respectively. These configurations are described in detail in Section 6. The approach presented here is motivated by the observations recently used to obtain optimal lower bounds on all higher moments of the electric field for two-phase random dielectrics, see Lipton (2004).

It is pointed out that the lower bounds on the moments (1.3) and L^∞ norms (1.4) given in Section 4 can be applied directly to the strain based analysis of failure initiation in composites (Asp, Berglund and Talerja, 1996; Gosse and Christensen, 2001). For example the lower bounds on the L^∞ norms (1.4) given by (4.5), (4.6), (4.8), (4.9), for $r = \infty$, provide explicit conditions on the applied strain for which the local strain will lie out side the strength domain of the matrix phase inside a fiber-epoxy composite. For other composite systems requiring a Weibull-type failure analysis, the lower bounds on the moments (1.3) given by (4.5), (4.6), (4.8), (4.9) deliver lower bounds on the failure probability of the composite material.

Earlier investigations into the behavior of local fields inside elastic composites have identified the optimal inclusion shapes that minimize the maximum eigenvalue of the local stress for a given constant applied stress. These investigations are carried out in the context of two-phase linear elasticity. The work presented in Wheeler (1993) provides an optimal lower bound on the supremum of the maximum principle stress for a single simply connected stiff inclusion in an infinite matrix subject to a remote stress at infinity. The optimal shapes are given by ellipsoids. The work presented in Grabovsky and Kohn (1995) provides an optimal lower bound on the supremum of the maximum principle stress for two-dimensional periodic composites consisting of a single simply connected stiff inclusion in the period cell. The bound is given in terms of the area fraction of the included phase and for an explicit range of prescribed average stress the optimal inclusions are given by Vigdergauz (1994) shapes.

2 Mathematical formulation of the problem.

The composite is contained inside a cube Q and no constraints are placed upon the arrangement of the two materials inside Q. The volume fractions of materials one and two are denoted by θ_1 and θ_2 respectively. It is supposed that Q is the period cell for an infinite periodic medium.

The elastic stress and strain fields $\sigma(\mathbf{x})$ and $\epsilon(\mathbf{x})$ inside the two-phase material satisfy $\epsilon_{ij}(\mathbf{x}) = (u_{i,j}(\mathbf{x}) + u_{j,i}(\mathbf{x}))/2$ and $\sigma(\mathbf{x}) = C(\mathbf{x})\epsilon(\mathbf{x})$. Here $C(\mathbf{x})$ is the local elasticity tensor and $u_{i,j}$ is the derivative of the i^{th} component of the displacement along the j^{th} direction. The elasticity tensor of materials one and two are specified by the shear and bulk moduli μ^1, κ^1 and μ^2, κ^2 respectively. With out loss of generality it is supposed that $\mu^1 > \mu^2$. The equation of elastic equilibrium inside each phase is given by

$$\mathrm{div}\,\sigma = 0. \tag{2.1}$$

It is assumed that there is perfect contact between the materials so that the displacement $\mathbf{u}$ and traction $\sigma\mathbf{n}$ are continuous across the two phase interface, i.e.,

$$\begin{aligned} \mathbf{u}_{|_1} &= \mathbf{u}_{|_2}, \\ \sigma_{|_1}\mathbf{n} &= \sigma_{|_2}\mathbf{n}. \end{aligned} \tag{2.2}$$

Here $\mathbf{n}$ is the unit normal to the interface pointing into material 2 and the subscripts indicate the side of the interface that the displacement and traction fields are evaluated on. The volume average of a quantity q over the cube Q is denoted by $< q >$ and for two-dimensional elastic problems the domain Q becomes the unit square and volume averages are replaced by area averages. The average strain field $< \epsilon >$ satisfies $< \epsilon >= \overline{\epsilon}$ where $\overline{\epsilon}$ is an applied constant strain. Here $\overline{\epsilon}$ can be interpreted as the imposed macroscopic strain. The elastic displacement $\mathbf{u}$ inside the composite is such that the difference $\mathbf{u}_i(\mathbf{x}) - \overline{\epsilon}_{ij}\mathbf{x}_j$ is periodic on Q. The effective elastic tensor C^e relating the average stress to the imposed macroscopic strain is defined by

$$< \sigma >= C^e\overline{\epsilon}. \tag{2.3}$$

3 Lower bounds on the dilatational strain field and sufficient conditions for optimality.

In this Section we establish lower bounds on the dilatational strain field inside each material. Sufficient conditions are identified that guarantee that lower bounds are attained. These conditions are used to establish the optimality of the bounds presented in Section 4.

The fourth order identity is denoted by $\mathbf{I}$ and $\mathbf{I}_{ijkl} = 1/2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. The projection onto the dilatational part of the strain is denoted by $\mathbf{\Pi}^H$ and is given explicitly by

$$\mathbf{\Pi}^H_{ijkl} = \frac{1}{d}\delta_{ij}\delta_{kl}. \tag{3.1}$$

The projection onto the deviatoric part of the strain is given by $\mathbf{\Pi}^D = \mathbf{I} - \mathbf{\Pi}^H$. The isotropic elasticity tensor associated with each component material is written as

$$C^i = 2\mu^i\mathbf{\Pi}^D + d\kappa^i\mathbf{\Pi}^H, \text{ for } i = 1, 2, \tag{3.2}$$

where $d = 2$ for planar elastic problems and $d = 3$ for the three dimensional problem.

For any symmetric $d \times d$ strain field $\eta(\mathbf{x})$ defined on Q one has

$$< \chi_2(\mathbf{x})\mathbf{\Pi}^H(\epsilon(\mathbf{x}) - \eta(\mathbf{x})) : (\epsilon(\mathbf{x}) - \eta(\mathbf{x})) > \geq 0. \tag{3.3}$$

Setting η equal to a constant strain $\overline{\eta}$ one obtains

$$< \chi_2(\mathbf{x})\mathbf{\Pi}^H\epsilon(\mathbf{x}) : \epsilon(\mathbf{x}) > \geq 2\mathbf{\Pi}^H\overline{\eta} :< \chi_2(\mathbf{x})\epsilon(\mathbf{x}) > -\theta_2\mathbf{\Pi}^H\overline{\eta} : \overline{\eta}. \tag{3.4}$$

Optimizing over $\overline{\eta}$ gives

$$< \chi_2(\mathbf{x})\mathbf{\Pi}^H \epsilon(\mathbf{x}) : \epsilon(\mathbf{x}) > \; \geq \frac{1}{\theta_2}\mathbf{\Pi}^H < \chi_2(\mathbf{x})\epsilon(\mathbf{x}) >:< \chi_2(\mathbf{x})\epsilon(\mathbf{x}) > . \tag{3.5}$$

It now easily follows from (3.1) that

$$< \chi_2(\mathbf{x})|\frac{tr\{\epsilon(\mathbf{x})\}}{d}|^2 > \; \geq \frac{1}{\theta_2}| < \chi_2(\mathbf{x})\frac{tr\{\epsilon(\mathbf{x})\}}{d} > |^2. \tag{3.6}$$

Expanding (2.3) one obtains

$$C^e\overline{\epsilon} =< \left((C^1 + \chi_2(C^2 - C^1)\right)\epsilon(\mathbf{x}) > . \tag{3.7}$$

Rearranging terms and taking the trace gives

$$tr\{(C^2 - C^1)^{-1}(C^e - C^1)\overline{\epsilon}\} =< \chi_2(\mathbf{x})tr\{\epsilon(\mathbf{x})\} > . \tag{3.8}$$

From (3.6) one obtains

$$< \chi_2(\mathbf{x})|\frac{tr\{\epsilon(\mathbf{x})\}}{d}|^2 > \; \geq \frac{1}{d^2\theta_2}|tr\{(C^2 - C^1)^{-1}(C^e - C^1)\overline{\epsilon}\}|^2. \tag{3.9}$$

For p and q such that $p \geq 1$ and $1/p + 1/q = 1$, an elementary estimate gives

$$\theta_2^{1/q} < \chi_2(\mathbf{x})|\frac{tr\{\epsilon(x)\}}{d}|^{2p} >^{1/p} \geq \; < \chi_2(\mathbf{x})|\frac{tr\{\epsilon(\mathbf{x})\}}{d}|^2 > \tag{3.10}$$

and it follows that

$$< \chi_2(\mathbf{x})|\frac{tr\{\epsilon(x)\}}{d}|^{2p} >^{1/p} \geq \frac{\theta_2^{1/p}}{d^2\theta_2^2}|tr\{(C^2 - C^1)^{-1}(C^e - C^1)\overline{\epsilon}\}|^2, \tag{3.11}$$

for $1 \leq p \leq \infty$. From (3.8) one easily sees that the lower bound given by (3.11) is optimal when the dilatational strain field $tr\{\epsilon\}/d$ is constant inside material two.

Similar arguments give the lower bound

$$< \chi_1(\mathbf{x})|\frac{tr\{\epsilon(\mathbf{x})\}}{d}|^2 > \; \geq \frac{1}{d^2\theta_1}|tr\{(C^1 - C^2)^{-1}(C^e - C^2)\overline{\epsilon}\}|^2. \tag{3.12}$$

and it follows that

$$< \chi_1(\mathbf{x})|\frac{tr\{\epsilon(x)\}}{d}|^{2p} >^{1/p} \geq \frac{\theta_1^{1/p}}{d^2\theta_1^2}|tr\{(C^1 - C^2)^{-1}(C^e - C^2)\overline{\epsilon}\}|^2, \tag{3.13}$$

for $1 \leq p \leq \infty$. Here equality holds in (3.13) when the dilatational strain field is constant inside phase one.

4 Optimal lower bounds on the moments of dilatational strain field for composites subject to hydrostatic strain.

Optimal lower bounds on the moments and L^∞ norms of the dilatational strain field are presented. The bounds are given in terms of the volume fractions of materials one and two. The particular form of the lower bounds depends upon whether the elastic materials are well ordered, $\kappa^1 > \kappa^2$ or non well ordered $\kappa^1 < \kappa^2$.

We introduce

$$L^1 = \frac{\kappa^2 + 2\mu^2(d-1)/d}{\theta_1\kappa^2 + \theta_2\kappa^1 + 2\mu^2(d-1)/d}, \tag{4.1}$$

$$L^2 = \frac{\kappa^1 + 2\mu^1(d-1)/d}{\theta_1\kappa^2 + \theta_2\kappa^1 + 2\mu^1(d-1)/d}, \tag{4.2}$$

$$M^1 = \frac{\kappa^2 + 2\mu^1(d-1)/d}{\theta_1\kappa^2 + \theta_2\kappa^1 + 2\mu^1(d-1)/d}, \tag{4.3}$$

and

$$M^2 = \frac{\kappa^1 + 2\mu^2(d-1)/d}{\theta_1\kappa^2 + \theta_2\kappa^1 + 2\mu^2(d-1)/d}. \tag{4.4}$$

The lower bounds and the associated optimal configurations are presented in the following two subsections.

4.1 Optimal lower bounds for the well ordered case $\kappa^1 > \kappa^2$

For the well ordered case $\kappa^1 > \kappa^2$ and $L^1 \leq 1 \leq L^2$. The optimal lower bounds on the moments of the dilatational strain are given by the following results.

Optimal lower bounds on the moments of the dilatational strain in material one. For fixed values of θ_1 and θ_2 and imposed macroscopic strain $\epsilon_0 I$ the dilatational strain field $tr\{\epsilon(\mathbf{x})\}/d$ inside material one satisfies

$$|\epsilon_0|\theta_1^{1/r} L^1 \leq \; < \chi_1 |tr\{\epsilon(\mathbf{x})\}/d|^r >^{1/r}, \text{ for } 2 \leq r \leq \infty. \tag{4.5}$$

Moreover for $d = 2(3)$ the dilataional strain inside material one for the coated cylinder (sphere) assemblage with core of material one and coating of material two attains the lower bound (4.5) for every r in $2 \leq r \leq \infty$.

Optimal lower bounds on the moments of the dilatational strain in material two. For fixed values of θ_1, θ_2, and imposed macroscopic strain $\epsilon_0 I$ the dilatational strain field $tr\{\epsilon(\mathbf{x})\}/d$ inside material two satisfies

$$|\epsilon_0|\theta_2^{1/r} L^2 \leq \; < \chi_2 |tr\{\epsilon(\mathbf{x})\}/d|^r >^{1/r}, \text{ for } 2 \leq r \leq \infty. \tag{4.6}$$

Moreover for $d = 2(3)$ the dilatational strain inside material two for the coated cylinder (sphere) assemblage with core of material two and coating of material one attains the lower bound (4.6) for every r in $2 \leq r \leq \infty$.

Optimal lower bound on the L^∞ norm of the dilatational strain. For fixed values of θ_1, θ_2, and imposed macroscopic strain $\epsilon_0 I$ the dilatational strain field $tr\{\epsilon(\mathbf{x})\}/d$ satisfies

$$|\epsilon_0| L^2 \leq \| |tr\{\epsilon(\mathbf{x})\}/d| \|_{L^\infty(Q)}. \tag{4.7}$$

Moreover for $d = 2(3)$ the dilatational strain inside the coated cylinder (sphere) assemblage with core of material two and coating of material one attains the lower bound (4.7).

4.2 Optimal lower bounds for the non well ordered case $\kappa^1 < \kappa^2$

For $\kappa^1 < \kappa^2$ one has that $M^2 \leq 1 \leq M^1$. The optimal lower bounds on the moments of the dilatational strain are given by the following results.

Optimal lower bounds on the moments of the dilatational strain in material one. For fixed values of θ_1 and θ_2 and imposed macroscopic strain $\epsilon_0 I$ the dilatational strain field $tr\{\epsilon(\mathbf{x})\}/d$ inside material one satisfies

$$|\epsilon_0|\theta_1^{1/r} M^1 \quad \leq \quad < \chi_1 |tr\{\epsilon(\mathbf{x})\}/d|^r >^{1/r}, \text{ for } 2 \leq r \leq \infty. \tag{4.8}$$

Moreover for $d = 2(3)$ the dilatational strain inside material one for the coated cylinder (sphere) assemblage with core of material two and coating of material one attains the lower bound (4.8) for every r in $2 \leq r \leq \infty$.

Optimal lower bounds on the moments of the dilatational strain in material two. For fixed values of θ_1, θ_2, and imposed macroscopic strain $\epsilon_0 I$ the dilatational strain field $tr\{\epsilon(\mathbf{x})\}/d$ inside material two satisfies

$$|\epsilon_0|\theta_2^{1/r} M^2 \quad \leq \quad < \chi_2 |tr\{\epsilon(\mathbf{x})\}/d|^r >^{1/r}, \text{ for } 2 \leq r \leq \infty. \tag{4.9}$$

Moreover for $d = 2(3)$ the dilatational strain inside material two for the coated cylinder (sphere) assemblage with core of material one and coating of material two attains the lower bound (4.9) for every r in $2 \leq r \leq \infty$.

Optimal lower bound on the L^∞ norm of the dilatational strain. For fixed values of θ_1, θ_2, and imposed macroscopic strain $\epsilon_0 I$ the dilatational strain field $tr\{\epsilon(\mathbf{x})\}/d$ satisfies

$$|\epsilon_0| M^1 \quad \leq \quad \| |tr\{\epsilon(\mathbf{x})\}/d| \|_{L^\infty(Q)}. \tag{4.10}$$

Moreover for $d = 2(3)$ the dilatational strain inside the coated cylinder (sphere) assemblage with core of material two and coating of material one attains the lower bound (4.10).

5 Derivation of the lower bounds

In this Section we use the lower bounds given by (3.11) and (3.13) to derive the lower bounds presented in Section 4. Recalling that $\bar{\epsilon} = \epsilon_0 I$ and using (3.2) one easily calculates that the right hand side of (3.11) is given by

$$\frac{\theta_2^{1/p}}{d^2\theta_2^2} |tr\{(C^2 - C^1)^{-1}(C^e - C^1)\bar{\epsilon}\}|^2 = \frac{\theta_2^{1/p}}{d^2\theta_2^2} |\epsilon_0|^2 \frac{|d^{-1}C^e I : I - d\kappa^1|^2}{|\kappa^1 - \kappa^2|^2}, \tag{5.1}$$

where $C^e I : I = C^e_{ijkl}\delta_{ij}\delta_{kl}$. Similarly the right hand side of (3.13) is given by

$$\frac{\theta_1^{1/p}}{d^2\theta_1^2} |tr\{(C^1 - C^2)^{-1}(C^e - C^2)\bar{\epsilon}\}|^2 = \frac{\theta_1^{1/p}}{d^2\theta_1^2} |\epsilon_0|^2 \frac{|d^{-1}C^e I : I - d\kappa^2|^2}{|\kappa^1 - \kappa^2|^2}. \tag{5.2}$$

One has the bounds on the contraction $C^e I : I$ given by Kantor and Bergman (1984)

$$d^2\kappa^-_{HS} \leq \quad C^e I : I \quad \leq d^2\kappa^+_{HS}, \tag{5.3}$$

where κ_{HS}^{-} and κ_{HS}^{+} are the Hashin and Shtrikman (1963) bulk modulus bounds given by

$$\kappa_{HS}^{+} = \kappa^1\theta_1 + \kappa^2\theta_2 - \left(\frac{\theta_1\theta_2(\kappa^2-\kappa^1)^2}{\kappa^1\theta_2 + \kappa^2\theta_1 + 2\frac{d-1}{d}\mu^1}\right) \tag{5.4}$$

and

$$\kappa_{HS}^{-} = \kappa^1\theta_1 + \kappa^2\theta_2 - \left(\frac{\theta_1\theta_2(\kappa^2-\kappa^1)^2}{\kappa^1\theta_2 + \kappa^2\theta_1 + 2\frac{d-1}{d}\mu^2}\right). \tag{5.5}$$

Kantor and Bergman (1984) point out that the the bounds given by (5.3) hold both for the well ordered case $\kappa^1 > \kappa^2$, $\mu^1 > \mu^2$ and the non-well ordered case $\kappa^1 < \kappa^2$, $\mu^1 > \mu^2$.

For the well ordered case $\kappa^1 > \kappa^2$ one applies the inequality (5.3) to (5.1) and (5.2) to find that

$$\frac{\theta_2^{1/p}}{d^2\theta_2^2}|tr\{(C^2-C^1)^{-1}(C^e-C^1)\bar{\epsilon}\}|^2 \geq \frac{\theta_2^{1/p}}{\theta_2^2}|\epsilon_0|^2\frac{|\kappa_{HS}^{+}-\kappa^1|^2}{|\kappa^1-\kappa^2|^2} = \theta_2^{1/p}|\epsilon_0|^2(L^2)^2 \tag{5.6}$$

and

$$\frac{\theta_1^{1/p}}{d^2\theta_1^2}|tr\{(C^1-C^2)^{-1}(C^e-C^2)\bar{\epsilon}\}|^2 \geq \frac{\theta_1^{1/p}}{\theta_1^2}|\epsilon_0|^2\frac{|\kappa_{HS}^{-}-\kappa^2|^2}{|\kappa^1-\kappa^2|^2} = \theta_1^{1/p}|\epsilon_0|^2(L^1)^2 \tag{5.7}$$

and the bounds (4.5) and (4.6) now follow easily from (3.11) and (3.13).

To obtain (4.7) we recall (4.6) for $r=\infty$ to see that

$$|\epsilon_0|L^2 \leq \quad \||tr\{\epsilon(\mathbf{x})\}/d|\|_{L^\infty(Q_2)} \quad \leq \||tr\{\epsilon(\mathbf{x})\}/d|\|_{L^\infty(Q)} \tag{5.8}$$

and (4.7) follows.

For the non-well ordered case $\kappa^1 < \kappa^2$ one applies the inequality (5.3) to (5.1) and (5.2) to find that

$$\frac{\theta_2^{1/p}}{d^2\theta_2^2}|tr\{(C^2-C^1)^{-1}(C^e-C^1)\bar{\epsilon}\}|^2 \geq \frac{\theta_2^{1/p}}{\theta_2^2}|\epsilon_0|^2\frac{|\kappa_{HS}^{-}-\kappa^1|^2}{|\kappa^1-\kappa^2|^2} = \theta_2^{1/p}|\epsilon_0|^2(M^2)^2 \tag{5.9}$$

and

$$\frac{\theta_1^{1/p}}{d^2\theta_1^2}|tr\{(C^1-C^2)^{-1}(C^e-C^2)\bar{\epsilon}\}|^2 \geq \frac{\theta_1^{1/p}}{\theta_1^2}|\epsilon_0|^2\frac{|\kappa_{HS}^{+}-\kappa^2|^2}{|\kappa^1-\kappa^2|^2} = \theta_1^{1/p}|\epsilon_0|^2(M^1)^2 \tag{5.10}$$

and the bounds (4.8) and (4.9) now follow easily from (3.11) and (3.13).

To obtain (4.10) we recall (4.8) for $r=\infty$ to see that

$$|\epsilon_0|M^1 \leq \quad \||tr\{\epsilon(\mathbf{x})\}/d|\|_{L^\infty(Q_1)} \quad \leq \||tr\{\epsilon(\mathbf{x})\}/d|\|_{L^\infty(Q)} \tag{5.11}$$

and (4.10) follows.

6 Optimality

In this Section it is shown that the lower bounds presented in Section 4 are attained by the dilatational strain fields inside the Hashin–Shtrikman (1962) coated sphere and cylinder assemblages. The coated cylinder assemblage is constructed as follows. A space filling configuration of disks of different sizes ranging down to the infinitesimal is placed inside the unit square Q. Each disk is then partitioned into an annulus called the coating and a concentric disk called the core. The area fractions of coating and core are the same for all disks. The unit square Q filled with the coated cylinder assemblage is illustrated in Figure 1. The construction of the coated sphere assemblage follows the same pattern. A space filling configuration of spheres is placed inside the unit cube. Each sphere is partitioned into a spherical shell called the coating and a concentric sphere called the core. Here the volume fractions of coating and core are the same for every sphere.

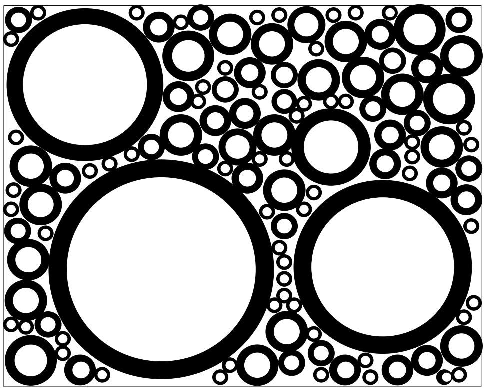

Figure 1: The unit square is filled with the Hashin – Shtrikman coated cylinder assemblage.

The explicit formula for the bulk modulus for the coated spheres construction was derived in Hashin (1962). The formula for the bulk modulus for the coated cylinders construction was given by Hashin and Rosen (1964). It is well known from the work of Hashin and Shtrikman (1963) that the associated effective bulk moduli for the coated sphere assemblages attain the bulk modulus bounds κ_{HS}^{-} and κ_{HS}^{+}. The analogous statements for the coated cylinder assemblages can be found in the work of Hashin and Rosen (1964). It is pointed out that the dilatational strain fields are constant inside the core phase and inside the coating phase for the coated sphere and cylinder assemblages. These observations together with the optimality conditions presented in Section 2 and the bounds given in Section 4 indicate that that the dilatational strain inside the coated sphere and cylinder assemblages are extremal.

For reference we list the effective bulk moduli and dilatational strain fields for the coated sphere and cylinder assemblages. The dilatational strain fields are computed for an imposed hydrostatic strain given by $\epsilon_0 I$. For assemblages with core of material one and coating of material two the effective bulk modulus is given by κ_{HS}^{-}, the dilatational strain field inside the core is given by $\epsilon_0 L^1$ and the dilatational strain field inside the coating is given by $\epsilon_0 M^2$. For assemblages with core of material two and coating of material one the effective bulk modulus is given by κ_{HS}^{+}, the dilatational strain field inside the core is given by $\epsilon_0 L^2$ and the dilatational strain field inside the coating is given by $\epsilon_0 M^1$. From these observations it is evident that the bounds (4.5), (4.6), (4.8) and (4.9) are attained by the dilatational strain fields inside the coated sphere and coated cylinder assemblages. For $\kappa^1 > \kappa^2$ one checks that $L^2 > M^1$ and it follows that coated sphere and cylinder assemblages with a core of material two and coating of material one have dilatational strains that attain the lower bound (4.7). For $\kappa^1 < \kappa^2$ one checks that $L^2 < M^1$ and it follows that coated sphere and cylinder assemblages with a core of material two and coating of material one have dilatational strains that attain the lower bound (4.10).

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