The Binomial Distribution

Learning Goals.

1. *Know how to compute the binomial coefficients and know what they mean.*

   a. The binomial coefficient $\binom{n}{k}$ (pronounced "n choose k") is the number

$$\frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \cdots \cdot \frac{n-k+1}{k}.$$

Example.

$$\text{"17 choose 6"} = \binom{17}{6} = \frac{17}{1} \cdot \frac{16}{2} \cdot \frac{15}{3} \cdot \frac{14}{4} \cdot \frac{13}{5} \cdot \frac{12}{6} = 12{,}376.$$

Example.

$$\binom{5}{0} = 1; \ \binom{5}{1} = 5; \ \binom{5}{2} = 10; \ \binom{5}{3} = 10; \ \binom{5}{4} = 5; \ \binom{5}{5} = 1.$$

   b. The binomial coefficient $\binom{n}{k}$ is the number of ways that exactly k heads can occur in a sequence of n coin flips.

Example. If a coin is flipped 5 times then there will be $\binom{5}{2}$ ways that two heads can occur. $\binom{5}{2} = 10$, so there are 10 ways for 2 heads to occur.[1]

Example. For small n, we can depict all the binomial coefficients as actual counts. For example, consider what may happen in 4 flips. All the possible outcomes are listed in the table below, and they have been sorted into columns by the number of heads occurring. The height of the column over k is $\binom{4}{k}$.

Table 1. The meaning of the binomial coefficients $\binom{n}{k}$ with $n = 4$.

	6			HHTT			
	5			HTHT			
$\binom{4}{k}$	4		HTTT	HTTH	HHHT		
	3		THTT	THHT	HHTH		
	2		TTHT	THTH	HTHH		
	1	TTTT	TTTH	TTHH	THHH	HHHH	
		0	1	2	3	4	$\leftarrow$ k

[1] Note that we did not absolutely have to use the binomial coefficient, since we could have counted directly: 1) HHTTT, 2) HTHTT, 3) HTTHT, 4) HTTTH, 5) THHTT, 6) THTHT, 7) THTTH, 8) TTHHT, 9) TTHTH, 10) TTTHH. However, the formula saves time and avoids the errors that could come from a miscount. Moreover, for larger numbers the direct counting method would not be practical. No one would have the patience or bookkeeping skill needed to determine *by writing out all the possibilities* that in 17 flips there are 12, 376 ways to get 6 heads. For this, we really need the formula.

2. *Know how to represent and solve probability problems such as the following.*
Problem. A fair coin is flipped 12 times. *a*) What is the probability of getting exactly 6 heads? *b*) At least 6 heads? *c*) Between 4 and 8 heads?
Solution. There are $2^{12} = 4096$ ways of flipping 12 coins, and they are all equally likely. Therefore, we can calculate the probabilities of events by counting the number of outcomes in each event.
   a) The number of ways of flipping 6 heads in 12 flips is $\binom{12}{6} = 924$. Thus, the probability of flipping exactly 6 heads is 924/4096, or approximately 0.226.
   b) The number of ways of flipping *at least* 6 heads in 12 flips is

$$\binom{12}{6} + \binom{12}{7} + \binom{12}{8} + \binom{12}{9} + \binom{12}{10} + \binom{12}{11} + \binom{12}{12}.$$

   A little calculation shows that this is 924+792+495+220+66+12+1 which adds up to 2510. The probability, therefore, is 2510/4096, or approximately 0.613.
   c) The number of ways of flipping between 4 and 8 heads in 12 flips is

$$\binom{12}{4} + \binom{12}{5} + \binom{12}{6} + \binom{12}{7} + \binom{12}{8} = 495 + 792 + 924 + 792 + 495 = 3498.$$

   The probability of this is 3498/4096, or approximately 0.854.
3. *Know how to represent and solve probability problems, such as the following, involving trials where the outcomes are not equally probable.*
Problem. A very large jar contains millions of red and green beads thoroughly mixed in the ratio 5 green to 3 red. (We are assuming that there are so many beads that the removal of a dozen or fewer does not affect the probability of drawing a red or a green bead.) Twelve beads are drawn from the jar. *a*) What is the probability of getting exactly 6 green beads? *b*) Exactly 7 green beads? *c*) Between 4 and 8 green beads? *d*) Eight or more green beads?
There is a discussion following the solution that explains the rationale. You may wish to read the solution first and try to make sense of it. Or you may prefer to read the discussion first.
Solution. The probability of drawing a green bead is 5/8. Therefore, by the binomial formula,
   a) the probability of drawing exactly 6 green beads is

$$\left(\frac{5}{8}\right)^6 \left(\frac{3}{8}\right)^6 \binom{12}{6} = 0.153158\ldots$$

   and
   b) the probability of drawing exactly 7 green beads is

$$\left(\frac{5}{8}\right)^7 \left(\frac{3}{8}\right)^5 \binom{12}{7} = 0.218797\ldots.$$

Caution! Notice that in the formulae above, both fractions—such as $\left(\frac{5}{8}\right)$—and binomial coefficients—such as $\binom{12}{6}$—occur. Do not confuse the two.

c) The probability of drawing between 4 and 8 green beads is

$$\left(\tfrac{5}{8}\right)^4\left(\tfrac{3}{8}\right)^8\tbinom{12}{4}+\left(\tfrac{5}{8}\right)^5\left(\tfrac{3}{8}\right)^7\tbinom{12}{5}+\left(\tfrac{5}{8}\right)^6\left(\tfrac{3}{8}\right)^6\tbinom{12}{6}+\left(\tfrac{5}{8}\right)^7\left(\tfrac{3}{8}\right)^5\tbinom{12}{7}+\left(\tfrac{5}{8}\right)^8\left(\tfrac{3}{8}\right)^4\tbinom{12}{8}$$

Rounding to the nearest millionth, this works out to:

$$0.029538 + 0.078767 + 0.153158 + 0.218797 + 0.227914 \cong 0.70817$$

d) The probability of drawing 8 or more green beads is

$$\left(\tfrac{5}{8}\right)^8\left(\tfrac{3}{4}\right)^4\tbinom{12}{8}+\left(\tfrac{5}{8}\right)^9\left(\tfrac{3}{8}\right)^3\tbinom{12}{9}+\left(\tfrac{5}{8}\right)^{10}\left(\tfrac{3}{8}\right)^2\tbinom{12}{10}+\left(\tfrac{5}{8}\right)^{11}\left(\tfrac{3}{8}\right)^1\tbinom{12}{11}+\left(\tfrac{5}{8}\right)^{12}\left(\tfrac{3}{8}\right)^0\tbinom{12}{12}$$

Rounding to the nearest millionth, this works out to:

$$0.227914 + 0.168825 + 0.084412 + 0.025580 + 0.003553 \cong 0.51028$$

Discussion. There are $2^{12} = 4096$ ways of drawing 12 beads, each designated by a sequence such as *RRGRGGGRRGRR* or *RGRRGGGRGRRG*, *etc.* However, the different outcomes in this case are **not** equally likely. For example, since the probability of any one bead being red is 3/8, the probability of drawing 12 red beads in a row is $(3/8)^{12}$, or about 8 in a million. On the other hand, the probability of drawing 12 green beads in a row is $(5/8)^{12}$, or about 3553 in a million. Because the probabilities of different outcomes are different, we **cannot** calculate the probabilities of events simply by counting the number of outcomes in each event and then dividing by the total number of outcomes.

The important thing, the thing that makes it possible for us to calculate the probabilities, is that any two outcomes *with the same number of greens* have equal probability. For example, any two outcomes with 5 greens have the same probability. Why is this? Let's do an example that will illustrate the reasoning. The the probability of the draw *RRGRGGGRRGRR* is:

$$\frac{3}{8}\cdot\frac{3}{8}\cdot\frac{5}{8}\cdot\frac{3}{8}\cdot\frac{5}{8}\cdot\frac{5}{8}\cdot\frac{5}{8}\cdot\frac{3}{8}\cdot\frac{3}{8}\cdot\frac{5}{8}\cdot\frac{3}{8}\cdot\frac{3}{8}=\left(\frac{5}{8}\right)^5\left(\frac{3}{8}\right)^7=0.00009945\ldots.$$

This is because the first ball must be red (and there is a 3/8 chance of that), the second must be red (3/8), the third must be green (5/8), *etc.* The probability of the draw *RRRGGGGGRRRR*—which also has 5 greens and 7 reds—is:

$$\frac{3}{8}\cdot\frac{3}{8}\cdot\frac{3}{8}\cdot\frac{5}{8}\cdot\frac{5}{8}\cdot\frac{5}{8}\cdot\frac{5}{8}\cdot\frac{5}{8}\cdot\frac{3}{8}\cdot\frac{3}{8}\cdot\frac{3}{8}\cdot\frac{3}{8}=\left(\frac{5}{8}\right)^5\left(\frac{3}{8}\right)^7=0.00009945\ldots.$$

This shows how to calculate the probability of any one outcome in the event (technical word!) of drawing 5 greens. We can calculate the number of outcomes in this event by using the binomial coefficient. In this case, it's $\binom{12}{5} = 792$. We get the probability of the event "drawing 5 greens" by adding together the probabilities of all the outcomes in the event. In this case, all 792 outcomes in the event have the same probability, so the probability of the event is

$$\left(\frac{5}{8}\right)^5\left(\frac{3}{8}\right)^7\binom{12}{5}\cong(0.00009945)\cdot 792\cong 0.0788.$$