A. The picture above shows a portion of the unit circle in the x-y-plane, as well as a ray OB that makes a positive angle of θ radians with the positive x-axis.

1) What are the lengths of the following segments: OB, OD, BD, DA, AC?

2) What are the coordinates of the following points: B, C, D?

3) What is the area of triangle OAB?

4) What is the area of sector OAB? (Hint: This sector makes up a fraction of the circle equal to $\frac{\theta}{2\pi}$, while the area of the circle is π.)

5) By comparing the areas of triangle OAB and sector OAB, show that $0 < \sin\theta < \theta$ when θ is a small positive angle.

6) READ the Squeeze Theorem on page 99. What does 5) imply about $\lim_{\theta\to 0^+} \sin\theta$? Is the sine function continuous from the right at 0?

7) Is it continuous at 0?

8) By comparing the length of segment DA with the length of segment DB, show that $1 - \theta < \cos\theta < 1$ for small positive θ.

9) What does this imply about $\lim_{\theta\to 0^+} \cos\theta$?

10) Is the cosine function continuous at 0?

Morals: a) You got some experience with a representation of the sine and cosine in the unit circle. b) You saw that sine and cosine are both continuous at 0. c) You learned to use the Squeeze Theorem.

B. Now, we are going to show that the sine and the cosine are continuous everywhere.

1) Using the angle addition formula express $\sin(a+h)$ in terms of the sine and cosine of a and h.

2) True or false: $\lim_{x\to a} f(x)$ exists if and only if $\lim_{h\to 0} f(a+h)$ does, and if the limits exist, they are equal.

3) Using 1), 2) and the limit laws (addition, multiplication, etc.), show that $\lim_{\theta\to a} \sin\theta = \sin a$. Thus, conclude that the sine is continuous everywhere.

4) Show that the cosine is continuous everywhere.

Morals: a) You reviewed the important angle addition formulae. b) You saw how to express any limit as a limit at 0. c) You saw that sine and cosine are continuous.

C. Now, we return to the diagram to explore the functions $\frac{\sin x}{x}$ *and* $\frac{1-\cos x}{x}$.

1) Referring to your work in class yesterday, show that $\sin\theta \le \theta \le \frac{\sin\theta}{\cos\theta}$.

2) From this, deduce that $\cos\theta \le \frac{\sin\theta}{\theta} \le 1$.

3) Using the Squeeze Theorem, find the value of $\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta}$?

4) Use the fact that $\sin(-\theta) = -\sin\theta$ to show that $\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1$. Conclude $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

5) Show that $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0$. (See hint on page 103.)

Moral: You have made a rigorous demonstration of two important limits. We'll use them!

D. We are going to find the instantateous rate of change of the sine and cosine.

1) Find the average rate of change of sin over the interval $[\alpha, \alpha+\beta]$.

2) Find the limit of "the average rate of change of sin on $[\alpha, \alpha+\beta]$" as $\beta \to 0$. You will need to use the angle addition formula, some algebra, the limits rules and the two remarkable limits above.

3) Do the same as in 1) and 2) for the cosine.

Moral: You calculated the derivatives of the sine and the cosine! You're weeks ahead of schedule! Buy youself a fine dinner and enjoy your accomplishment.