Proof, reasoning and imagination

In research mathematics, rigorous proof is the means by which new results are validated and is the standard form for reporting findings and for compiling and recording mathematical knowledge. The Greek model of a deductive science, as exemplified by Euclid, is paradigmatic.

Rigorous deduction is also a working tool for research mathematicians. In exploring an unfamiliar topic, hunches and conjectures are checked by constructing proofs—or attempting to do so. Moreover, the search for proof can lead to new conjectures and deeper understanding.

All this taken for granted, few mathematicians would regard proof as the sole or perhaps even the chief instrument of discovery and invention. Proof has a unique role in the discipline of mathematics, but it works hand-in-hand with experience and experimentation (possibly by computer), inductive reasoning, generalization, analogy, intuition, mental imagery and mysterious unconscious processes. The best mathematics seems to come from a marriage of discipline and freedom, when rigorous logical structures serve as a medium for the imagination.

The role of proof in the classroom, in contrast, generates contoversy. Lynn Arthur Steen, in the concluding chapter of NCTM's 1999 Yearbook (which is devoted to mathematical reasoning), writes that "there is precious little agreement on how, when, why, or to whom to teach it."[S] While this statement refers primarily to the K-12 curriculum, even at the university level there is a range of opinion. Should freshman calculus include proofs? Of what? With what level of rigor? For what purposes or to what ends? (See [T].) Many popular American textbooks include careful, rigorous proofs, but they are dumped in appendices—as if to acknowledge their uncertain relevance to calculus learners. The same uncertainty persists in linear algebra, differential equations and other mathematics courses that are taken by large numbers of students headed into science, enineering or other technical professions. Only in advanced courses designed for those intending to become professional mathematicians is the place of proof perfectly clear.

Perhaps proof is to narrow an idea, too specialized in its uses, too stylized in its conventions, and too much governed by the needs arising from its professional uses to figure in the education of those not destined for careers in mathematics. If so, there is something more general and capable of more uses that can claim its place. Clearly, everyone should learn to *reason mathematically*. The idea covers vastly more ground than the idea of proof. To formulate generalizations and to create logical arguments that justify and explain them—these skills are useful whether the subject is mathematics or law or medicine or anything else, and to exercise them in a mathematical setting is a great part of what it means to be a responsible, reasonable math user. Moreover, as these skills mature by being subjected to higher and higher discipline, they blend into the activity of proving. Proof is simply a way of recording crucial pieces of reasoning so that they can be can be evaluated, shared and replicated.

This, at least, is what many claim. In particular, it is the position of the *Principles and Standards for School Mathematics* (NCTM, 2000). This document includes among it's ten standards, a strand for "Reasoning and Proof," and here we read:

> Mathematical reasoning and proof offer powerful ways of developing

> and expressing insights about a wide range of phenomena. People who reason and think analytically tend to note patterns, structure, or regularities in both real-world situations and symbolic objects; they ask if those patterns are accidental or if they occur for a reason; and they conjecture and prove. Ultimately, a mathematical proof is a formal way of expressing particular kinds of reasoning and justification.

Some additional excerpts from the same section confirm that the authors of the Standards understand by "mathematical reasoning" the process of finding true mathematical statements and providing justification in a form that meets accepted standards of evidence and logic. This, to them, is the essential nature of reasoning and proof:

> Doing mathematics involves discovery. Conjecture—that is, informed guessing—is a major pathway to discovery. ...
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> statements need to be supported or refuted by evidence ...
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> mathematical reasoning is based on specific assumptions and rules ...
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> [proofs are] arguments consisting of logically rigorous deductions of conclusions from hypotheses ...

As I already suggested, this sort of activity is not uniquely mathematical. A person who could note patterns and explain them clearly, persuasively and logically would be have an advantage in almost any intellectual task, and would be particularly well equipped to communicate and cooperate with others similarly engaged. This is a superb argument for including *reasoning* among the things that every student ought to learn. There is no question of its value.

But this brigs us to the brink of a very serious misunderstanding of what mathematics is all about. Go back, if you will, to the two paragraphs that introduced this section. In research mathematics, proof—in the sense of a well-articulated argument according to prescribed rules—is usually just the finishing touch on a discovery, the vehicle in which it is shared with the public, the final check on its truth, the means by which it gains acceptance in the mathematical community. The way that truths are *discovered* is entirely different. In 1968, Paul Halmos wrote:

> Mathematics—this may surprise you or shock you some—is never deductive in its creation. The mathematician at work makes vague guesses, visualizes broad generalizations, and jumps to unwarrented conclusions. He arranges and rearranges his ideas, and he becomes convinced of their truth long before he can write down a logical proof.

Similarly, Fields Medalist Richard Borcherds declared in 1998:

> The logical progression comes only right at the end, and it is in fact quite tiresome to check that all the details really work. Before that, you have to fit everything together by a lot of experimentation, guesswork, and intuition. (Both passages quoted in [K], page 269.)

The fact that in the end we frame things deductively must not lead to the mistaken opinion that deduction is the whole game. As I said before, deduction is a way of checking and verifying. The mathematical community demands it as a reliable test of truth, and it is the accepted form for commmunicating mathematical knowledge. Also, reasoning in the sense described in the *Standards*—making generalizations and fitting them into a logical framework—plays a role along the pathways to discovery. The experienced mathematician reasons easily and naturally, just as a skilled pianist uses technique, such as timing, voicing and ornamentation, with little conscious effort. This kind of reasoning, however, is only a condition for good mathematical thinking—not its apotheosis. When I say this, I hope to be understood in much the same way as had I said that good technique is a condition for good musicianship, not its highest realization. Indeed, the most brilliant technique is quite worthless without that essential something—call it musicianship or what you will—that we recognize so easily yet find so difficult to describe.

The way that mathematicians speak about the work they do has always been limited by the manners and conventions of the community and perhaps by personal inhibitions as well. Many mathematicians, like craftsmen, take pains to hide the joints and seams and rub away the tool marks. What they make public is the finished product, not the toil that went into the making. Gauss exhibited this tendency in the extreme, comparing himself to a builder who removes the scaffolding when the construction is complete. This makes sense if mathematics is a body of abstract knowledge transcending individual contributors, an intellectual palace that mathematicians are called upon to furnish. Likewise, those guiding and upholding the discipline naturally address the universal and impersonal features rather than the experiences of individuals. We pay attention to the final product, but say very little about the experience of making it. Yet, if mathematicians aim to share their way of thinking, or if they simply take into account the fact that each generation must pass its craft on to the next or else let it die, then there may be a reason to describe how mathematical work is experienced and how and why it fascinates.

What does it feel like to do mathematics? I cannot speak universally, even of my own experience. However, at times it can be very much like going into a mental studio of sorts, a place is furnished with abstract objects. Some are clear and distinct while others are vague or tentative in appearance. Some have no visual correlates at all, but are merely present in a ghostly way. Some are representatives of the actual things I'm thinking about—a triangle, for example—while others resemble the notation I use on paper. In various ways, the imagery registers or represents significant features of the mathematics. I can modify or distort these images, rearrange the parts, "zoom in" on features or examine them from different perspectives. When I am thinking hard about a mathematical problem, I often find myself trying to assemble imagery in ways that satify relevant constraints and meet the requirements of the problem. For example, I may try to verify a a conjecture by fitting it into a more complete or more detailed story or picture. (Only rarely do I try to put together stepwise logical arguments.) Perhaps most significant, the imagry does not become useful and effective until it has acquired familiarity.

The work of the imagination is difficult to describe. Introspection is largely disregarded by cognitive scientists, because there is never more than one observer, and his reports cannot be verified or checked. The account I give about what is happening in my mind may or may not have any relationship to the actual systems that enable me to think. Imaginary objects, however—whether it be mathematical or any other kind—are not necessarily private and hidden. Literary characters, like Tom Sawyer, are at once purely imaginary and completely public and intersubjective. You and I agree that Tom Sawyer, a clever lad, knew Huck Finn and Becky Thatcher. We do not know the exact color of his hair, or his height in feet and inches and are resigned never to know. There are yet other things about him that I do not know off hand but could determine by referring to the definitive treatise, where they might be given precisely (how long was he lost in the cave) or where enough information to make an informed guess might be available.

Mark Johnson's remarks about reasoning strike a very similar note:

> ... human rationality is imaginative through and through, insofar as it involves image-schematic structures that can be metaphorically projected from concrete to more abstract domains of understanding. Obviously, this is not imagination in the Romantic sense of unfettered creative fancy; rather, [it is] an extended Kantian view of imagination as a capacity for ordering mental representations into unified, coherent, meaningful wholes that we can understand and reason about. Imagination, in this sense, mediates between sense perception and our more abstractive conceptualizing capacities; it makes it possible to conceptualize various structural aspects of our experience and to formulate propositional descriptions of them. [J, page 198]

In the parts of my work that I feel are most creative, I deal directly with these mental objects, designing and creating them or putting them together from simpler mental objects according the requirements of the situation. The qualities of the thing must mark or represent the important features of the problem, question or situation. Getting this right may take a great deal of disciplined work, and involves studying and understanding the way that others have treated similar objects and what is known and written about their properties.

A narrative of sorts accompanies the imagery, somehow laying down rules or constraints that are not explicit on the mental stage. My characters may have histories or background files I have built up from textbook presentations, research papers and my own investigations. Often my work consists in revising the images over and over again to get them represent more accurately or completely what I know. Sometimes I forget or ignore the constraints that ought to apply, and then I might create a faulty image and leap to an incorrect conclusion. The narrative usually evolves with the imagery. When I reach a solution, or a conclusion or an insight, there is a sudden harmony in the imagery. In the mean time, I have also been modifying and revising the narrative, so when the harmony is achieved there is often along with it a fully formed outline for a proof. Sometimes I fool myself and seem to perceive a harmony that only evaporates as the narrative is worked through in detail. It's as if a couple of businessmen had sketched out a great deal that later fell apart as the details of the contract were being ironed out.

Building and playing with mental objects is something I can do voluntarily whenever I want to, but the imagery that matters—that leads to new knowledge, I mean—is not easy to make. When I am exploring something new, I may be overwhelmed by the complexity and unable pick out a mechanism or gimmick that enables me to find a meaning in it or extract a conclusion from it. Sometimes the consequences of the laws or rules that govern the situation are unclear or beyond my power to picture. I might have to work for days or weeks to incorporate enough insight into my conceptual model to get a structure that I can exploit. Sometimes it seems the goals are wrong, in the sense that the structures I build just grow more clumsy and ugly. I may decide to abandon the enterprise, or change directions. Sometimes there is something like a moment of grace, when a power beyond my understanding bestows an idea, or opens a vista to my internal eyes.

References

[K] Keith Devlin, *The Math Gene*, Basic Books, 2000.

[J] Mark Johnson, *The Body in the Mind : The Bodily Basis of Meaning, Imagination, and Reason*, University of Chicago Press, 1990.

[S] Lynn Arthur Steen, Twenty Questions about Mathematical Reasoning, in *Developing Mathematical Reasoning in Grades K-12.* Lee Stiff, Editor. Reston, VA: National Council of Teachers of Mathematics, 1999, pp. 270-285.

[T] Thomas Tucker, The role of proof in calculus courses, *Contemporary Issues in Mathematics Education.* E. A. Gavasto *et al.*, eds., MSRI Publications 36, Cambridge University Press, 1999.