Suppose f is a function with domain $[a,b]$. Since we will be concerend with subsets of $[a,b]$, we will simplify notation by letting $(p,q)^{\cdot}$ stand for $(p,q)\cap[a,b]$. If X is a subset of $[a,b]$,

$$f(X) := \{\, f(x) \mid x \in X \,\}.$$

Suppose $c \in [a,b]$. Recall what we mean when we say that f is continuous at c: For any real number $\epsilon > 0$, there is a real number $\delta > 0$ so that:

$$\text{for all } x \in [a,b],\ |x-c| < \delta \text{ implies } |f(x)-f(c)| < \epsilon.$$

The condition above (offset on its own line) is equivalent to:

$$\text{for all } x \in (c-\delta, c+\delta)^{\cdot},\ f(x) \in \big(f(c)-\epsilon, f(c)+\epsilon\big),$$

and to:

$$f\big((c-\delta, c+\delta)^{\cdot}\big) \subseteq \big(f(c)-\epsilon, f(c)+\epsilon\big).$$

Fact. If X is a bounded subset of the real line and u is the least upper bound of X, then any open interval that contains u also contains points of X. *Proof.* If u were surrounded by an open interval containing no ponts of X, there would be points in that interval to the left of u that were upper bounds for X. Then u could not be the least upper bount.

Boundedness Theorem. Suppose f is continuous on $[a,b]$. Then $f\big([a,b]\big)$ has an upper bound.

Proof. Since $a \in X$, X is nonempty. Since $X \subseteq [a,b]$, X is bounded above. Let u be the least upper bound of X. We claim that $u = b$. We prove this by an argument by contradiction. Suppose that $u < b$. By the definition of continuity (with $\epsilon = 1$), there is some $\delta > 0$ such that $f((u-\delta, u+\delta)^{\cdot}) \subseteq \big(f(u)-1, f(u)+1\big)$, i.e., $f((u-\delta, u+\delta)^{\cdot})$ is bounded above. Now by the Fact, $(u-\delta, u+\delta)^{\cdot}$ contains some points in X. We can conclude that $f([a, u+(\delta/2)])$ is bounded above. This contradicts the assumption that u is an upper bound of X. Thus, u is not strictly less than b, so $u = b$. /////

Maximum Value Theorem. Suppose f is continuous on $[a,b]$. Let M be the L.U.B. of $f([a,b])$. Then there is $c \in [a,b]$ such that $f(c) = M$.

Proof. If $f(a) = M$, we have nothing to prove, so suppose $f(a) < M$. Let X be the set of all points $x \in [a,b]$ such that $f([a,x]) \subseteq (-\infty, M)$. Since $a \in X$, X is not empty. Since $X \subseteq [a,b]$, X is bounded above. Let c be the least upper bound of X. We claim that $f(c) = M$. We prove this by an argument by contradiction. Suppose that $f(c) < M$. Let $\epsilon := (M - f(c))/2$. But by the definition of continuity, there is some $\delta > 0$ so that $\big(c-\delta, c+\delta)\big)$ is contained in $(f(c)-\epsilon, f(c)+\epsilon)$. Thus $f([a, c+\delta]) \subseteq (-\infty, M)$. This contradicts the assumption that c is an upper bound of X. /////