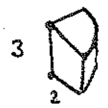
M2057 Test October 12, 2011 (3 pages) NAME: Answers

1. R is the region in the plane in the first quadrant and inside the disk of radius 2 about the origin. (It's shaped like a quarter pizza, with the point at the origin.)

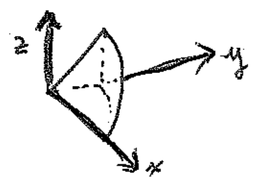
a) The following integral represents the volume of a solid: $\iint_R 3\,dA$. Describe that solid in words or draw a picture. Using basic geometry, compute its volume.

A quarter cake, 2 units in radius & 3 units high.



Volume = area base × height = $\pi \times 3 = 3\pi$

b) The following integral represents the volume of a solid: $\iint_R 2y\,dA$. Describe that solid in words or draw a picture. (Do not compute the volume.)



Piece of cake, top sliced at angle.

c) One of the following integrals represents the volume of the solid in part b).

$$i)\ \int_0^2 \int_0^{\sqrt{4-x^2}} 2y\,dx\,dy\,, \qquad ii)\ \int_0^2 \int_0^{\sqrt{4-x^2}} 2y\,dy\,dx\,, \qquad iii)\ \int_0^{\pi/2} \int_0^2 2r\sin\theta\,dr\,d\theta.$$

Which one is correct, and what is wrong with the others?

ii) is correct.

i) has wrong (or meaningless) order of integration

iii) has incorrect dA. Should be $\int_0^{\pi/2}\int_0^2 2r\sin\theta\, r\,dr\,d\theta$

2. Calculate the double integral:

$\iint_R \cos(x-y)\,dA,\ R = \{\,(x,y) \mid 0 \le x \le \pi/2,\ 0 \le y \le \pi/4\}$

$$\int_0^{\pi/2}\int_0^{\pi/4} \cos(x-y)\,dy\,dx = \int_0^{\pi/2} -\sin(x-y)\Big|_{y=0}^{y=\pi/4} dx$$

$$= \int_0^{\pi/2} -\sin\left(x-\tfrac{\pi}{4}\right) + \sin x\,dx = \cos\left(x-\tfrac{\pi}{4}\right) - \cos x\,\Big|_0^{\pi/2}$$

$$= \left[\cos\tfrac{\pi}{4} - \cos\tfrac{\pi}{2}\right] - \left[\cos -\tfrac{\pi}{4} - \cos 0\right]$$

$$= -\cos\tfrac{\pi}{2} + \cos 0 = 1$$

test continues...

3. Calculate the iterated integral: $\int_0^1 \int_0^{2y} \sqrt{1-y^2}\, dx\, dy$

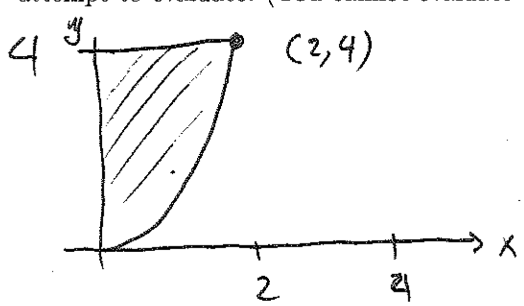
$$\int_0^1 x\sqrt{1-y^2}\Big|_0^{2y} dy = \int_0^1 2y\sqrt{1-y^2}\, dy = \int_1^0 -\sqrt{u}\, du$$

$u = 1-y^2$
$du = -2y\,dy$
$-du = 2y\,dy$
when $y=0$, $u=1$
when $y=1$, $u=0$

$$= \int_0^1 \sqrt{u}\, du = \frac{2}{3}u^{3/2}\Big|_0^1$$

$$= \frac{2}{3}$$

4) Sketch the region of integration for $\int_0^2 \int_{x^2}^4 f(x,y)\, dy\, dx$, and then rewrite the integral in the opposite order. Do not attempt to evaluate. (You cannot evaluate without knowing what $f(x,y)$ is, anyway.)



$$\int_0^4 \int_0^{\sqrt{y}} f(x,y)\, dx\, dy$$

5) Using polar coordinates, calculate the integral of $f(x,y) = x$ over the region that lies between the circle of radius 3 about the origin and the circle of radius 6 about the origin and is in the interior of the angle formed by the positive x-axis and the ray from $(0,0)$ through $(1/2, \sqrt{3}/2)$. (The measure of this angle is $\pi/3$.) (Polar coordinates are $x = r\cos\theta$, $y = r\sin\theta$ and $dA = r\, dr\, d\theta$.)

$$\int_0^{\pi/3} \int_3^6 r\cos\theta\, r\, dr\, d\theta = \int_0^{\pi/3} \cos\theta\, d\theta \int_3^6 r^2\, dr$$

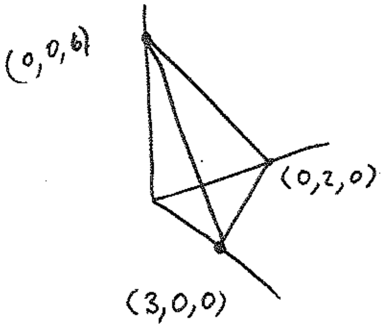
$$= \sin(\pi/3)\ \frac{r^3}{3}\Big|_3^6$$

$$= (\sin \pi/3)(63)$$

$$= \frac{\sqrt{3}}{2} \cdot 63$$

test continues...

6) Evaluate the triple integral: $\iiint_E x\,dA$, where E is the solid in the first octant $(0 \le x,\ 0 \le y,\ 0 \le z)$ bounded by the coordinate planes and the plane $z = -2x - 3y + 6$.



$$\int_0^3 \int_0^{-\frac{2}{3}x+2} \int_0^{-2x-3y+6} x\,dz\,dy\,dx =$$

$$\int_0^3 \int_0^{-\frac{2}{3}x+2} x(-2x-3y+6)\,dy\,dx = (*) =$$

see below

$$\int_0^3 \frac{2}{3}x^3 - 4x^2 + 6x\,dx = \frac{1}{6}x^4 - \frac{4}{3}x^3 + 3x^2 \Big|_0^3$$

$$= \left(\frac{1}{6}\cdot 9 - \frac{4}{3}\cdot 3 + 3\right)9 = \left(\frac{1}{2}\right)9 = \boxed{9/2}$$

$$(*)\ \int_0^{-\frac{2}{3}x+2} 6x - 2x^2 - 3xy\,dy = (6x-2x^2)y - \frac{3}{2}xy^2 \Big|_{y=0}^{y=-\frac{2}{3}x+2} = (6x-2x^2)\left(-\frac{2}{3}x+2\right) - \left(\frac{3}{2}x\right)\left(-\frac{2}{3}x+2\right)^2 =$$

$$= \left[(6x-2x^2) - \frac{3}{2}x\left(-\frac{2}{3}x+2\right)\right]\left(-\frac{2}{3}x+2\right) = \ldots = \frac{2}{3}x^3 - 4x^2 + 6x$$

7) An object in x-y-z-space fills the volume above the triangle with vertices $(1,1,0)$, $(3,1,0)$ and $(3,3,0)$ and below the surface $z = xy$. The density (mass per unit volume) at (x, y, z) is z. What is the total mass of the object?

$$\int_1^3 \int_1^x \int_0^{xy} z\,dz\,dy\,dx = \int_1^3 \int_1^x \frac{z^2}{2}\Big|_0^{xy} dy\,dx = \int_1^3 \int_1^x \frac{x^2y^2}{2}\,dy\,dx =$$

$$= \int_1^3 \frac{x^2}{2}\frac{y^3}{3}\Big|_{y=1}^{y=x} dx = \int_1^3 \frac{x^5}{6} - \frac{x^2}{6}\,dx = \frac{x^6}{36} - \frac{x^3}{18}\Big|_1^3$$

$$= \left(\frac{3^6}{36} - \frac{27}{18}\right) - \left(\frac{1}{36} - \frac{1}{18}\right) = \frac{729 - 54 - 1 + 2}{36}$$

$$= \boxed{\frac{676}{36} = \frac{169}{9} \approx 18.777}$$

$$\begin{array}{r} 27 \\ \times 27 \\ \hline 189 \\ 54 \\ \hline 729 \\ -53 \\ \hline 676 \end{array}$$

$$4\overline{)676} = 169$$
$$\begin{array}{r} 4 \\ \hline 27 \\ 24 \\ \hline 36 \end{array}$$

$$6x - 2x^2 - \frac{3}{2}x\left(-\frac{2}{3}x+2\right) =$$

$$= 6x - 2x^2 + x^2 - 3x$$

$$= 3x - x^2 = x(3-x)$$

$$x(3-x)\left(2 - \frac{2}{3}x\right) = \frac{2}{3}x(3-x)(3-x)$$

$$= \frac{2}{3}x(x^2 - 6x + 9)$$

$$= \frac{2}{3}x^3 - 4x^2 + 6x$$

END of TEST