**Test 3 (Take-Home) M2057 Spring 2004**

*Instructions.* Hand in on Friday May 7 before 5PM. Do not talk about this test with other students, use their work or show your work to others. You may talk about problems that do not appear on this test with others. You may ask me questions or seek my advice.

**1.** Suppose $F(x, y) = (P(x, y), Q(x, y))$. Define the words or phrases in italics, referring to $F$, $P$ or $Q$ as necessary.

**a.** $F$ is a *conservative vector field*

**b.** $F$ is an *exact vector field*

**c.** $F$ is a *path-independent vector field*

**2.** How are the properties of $F$ that are mentioned in the previous question related to one another? (Write a paragraph of 50 words or less. Write in full sentences and use correct grammar.)

**3.** (See Problem 33, page 1067.) **a.** Show that the vector field $F(x, y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ is exact at all points where it is defined. **b.** Show also that $F$ is **not** path-independent by calculating the path integral of $F$ around the unit circle. Show your work. **c.** Explain why this does not contradict what you said in 2.

**4.** Convert the following path integral to a double integral using Green's Theorem and then evaluate the double integral: $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = (y^6, xy^5)$ and $C$ is the ellipse $4x^2 + y^2 = 1$.

**5.** Find curl(curl$F$), where $F(x, y, z) = (\cos y\ \sin z, \sin x\ \cos z, \cos x\ \sin y)$

**6.** Let $b$ and $r$ be fixed constants. Suppose a fluid flows with velocity field $F(x, y, z) = (-y, x, bz)$. Set up and evaluate the flux integral that gives the net outward flow through the sphere $x^2+y^2+z^2 = r^2$. (Answer: $\frac{4}{3}\pi br^3$.)

**7.** Find the divergence of the vector field in Problem 6. Explain carefully how the Divergence Theorem allows us to express the flux integral in Problem 6 as as a volume integral. Over what volume must we integrate? How does this explain the answer to Problem 6?

**8** (Problem 32 on page 1121.) Use Stoke's Theorem to evaluate $\int\int_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z0 = (x^2yz, yz^2, z^3e^{xy})$, where $S$ is the part of the sphere $x^2 + y^2 + z^2 = 5$ that lies above the plane $z = 1$ and $S$ is oriented upwards.