Lecture 10. More examples

x.y:z. means Chapter x, section y, problem z.

3.5:33. A pair of dice is tossed 6 times. What is the probability that the sixth sum obtained is not a repetition?

If this is too hard, try an easier problem. Say a single die is tossed 3 times. What is the probability that the third number obtained is not a repetition?

Or try a more general problem. An experiment has a finite sample space $\Omega = \{\omega_1, \ldots, \omega_n\}$. If the experiment is performed m times, each trial being independent of the others, what is the probability that the last outcome has not previously occurred?

Solution to the the more general problem, as presented in class. First, note that there are two sample spaces involved. One is Ω. The other is the product Ω^m. An outcome in the latter is a sequence of m outcomes of Ω. Let $f : \Omega \to [0, 1]$ be the pmf of Ω. Then the probability of getting the outcome ω_i on any trial is $f(\omega_i)$, and the probability of *not* getting ω_i is $1 - f(\omega_i)$. The probability of missing ω_i in the first $m - 1$ trials (assuming they are independent of one another) is $(1 - f(\omega_i))^{m-1}$. Thus, the probability of "missing ω_i on the first $m - 1$ trials, and then getting it on the m^{th}"—which is an event in Ω^n—is

$$(1 - f(\omega_i))^{m-1} \cdot f(\omega_i).$$

Now, the event that the outcome of the last trial is not among the outcomes of the previous ones is the **disjoint union** of the events $E_i \subseteq \Omega^n$, where E_i is the event that outcome of the last trial is ω_i, occurring here for the first time. Thus, the probability we seek is:

$$\sum_{i=1}^{n} (1 - f(\omega_i))^{m-1} \cdot f(\omega_i). \qquad /////$$

3.5:35. If a coin is tossed, and then a random card is drawn from a deck (with replacement), what is the probability that a head is flipped before an ace is drawn?

Solution. Let H be the event of getting heads on first flip, let TN be the event of getting tails on first flip followed by a non-ace and let TA be the event of getting tails on first flip followed by an ace. Let E be the event of getting heads before an ace. Then

$$\begin{aligned} P(E) &= P(E|H)P(H) + P(E|TN)P(TN) + P(E|TA)P(TA) \\ &= P(H) + P(E)P(TN) + 0 \\ &= 1/2 + P(E)(1/2)(12/13) = 1/2 + P(E)(6/13). \end{aligned}$$

Thus, $P(E)(7/13) = 1/2$, so

$$P(E) = 13/14. \qquad /////$$

Extension a. An experiment has probability p of success, q of failure and $1 - p - q$ of neither. If independent trials are repeated over and over until either success or failure is obtained, what is the probability of an ultimate success?

Extension b. In a lottery with a very large number of tickets, 1 in 100 tickets pay \$2, 1 in 1000 tickets pay \$10, 1 in 10000 tickets pay \$50. What is a ticket worth? Now suppose that on Mardi Gras, 1 in ten tickets wins two additional tickets in the same lottery. What is a ticket on Mardi Gras worth? (Hint. Let $X(\omega_i)$ be the payoff for the outcome of getting a ticket of type i, where $i = 0$ if the ticket is a looser, $i = 1$ if it wins \$2, $i = 2$ if it wins \$10 and $i = 3$ if it wins \$50. Thus, $X(\omega_0) = 0$, $X(\omega_1) = 2$, etc. If $E(X) = e$, and ω_4 is the event of getting the bonus Mardi Gas ticket, then $X(\omega) = 2e$.)