Lecture 11. Expected Value (resumed)

To calculate EX, it is not necessary to treat each element of Ω individually. Elements may be grouped according to the values of X.

> Suppose that B is the set of all values of X, and let $[\![X = x]\!]$ denote the event $\{\, \omega \mid X(\omega) = x \,\}$. Then
>
> $$P(X = x) := P\big([\![X = x]\!]\big) = \sum_{\omega \in [\![X = x]\!]} f(\omega)$$
>
> Consequently,
>
> $$\begin{aligned} E(X) &= \sum_{\omega \in \Omega} X(\omega) f(\omega) \\ &= \sum_{x \in B} \Bigg(\sum_{\omega \in [\![X = x]\!]} x f(\omega) \Bigg) \\ &= \sum_{x \in B} x \Bigg(\sum_{\omega \in [\![X = x]\!]} f(\omega) \Bigg) \\ &= \sum_{x \in B} x P(X = x). \end{aligned}$$

In fact, in many applications, we need no more detailed knowledge of the sample space than what is provided by a random variable. If the sets $[\![X = x]\!]$ happen to contain many points, then often we may "clump them together" to view them as the single outcome $X = x$ without loosing any details relevant to our application. If this is possible, then the sample space can be identified with the set of all possible values x of X. Of course, each possible value x has a probability. This gives us a pmf $f(x) = P(X = x)$ for the random variable itself.

Example. *The hypergeometric random variable with parameters N, s, and n.* Let X be the number of special beads taken, if we draw n beads without replacement from a jar containing N beads, of which s are special. As we have shown previously,

$$P(X = x) = \frac{\binom{s}{x}\binom{N-s}{n-x}}{\binom{N}{n}}, \quad x = 0, 1, 2, \ldots s.$$

We can calculate the expectation as follows:

$$\begin{aligned} E(X) &= \sum_{x=0}^{s} x \frac{\binom{s}{x}\binom{N-s}{n-x}}{\binom{N}{n}} = \sum_{x=1}^{s} x \frac{\frac{s}{x}\binom{s-1}{x-1}\binom{N-s}{n-x}}{\frac{N}{n}\binom{N-1}{n-1}} \\ &= n \frac{s}{N} \sum_{x=1}^{s} \frac{\binom{s-1}{x-1}\binom{N-s}{n-x}}{\binom{N-1}{n-1}} \end{aligned}$$

Let us make the substitution $y = x - 1$ in the sum:

$$\sum_{x=1}^{s} \frac{\binom{s-1}{x-1}\binom{N-s}{n-x}}{\binom{N-1}{n-1}} = \sum_{y=0}^{s-1} \frac{\binom{s-1}{y}\binom{(N-1)-(s-1)}{n-1-y}}{\binom{N-1}{n-1}}.$$

Now, the terms on the right are the values of $P(Y = n)$, where Y is the hypergeometric random variable with parameters $N - 1$, $s - 1$ and $n - 1$. Hence, the sum is 1. It follows that

$$E(X) = n\frac{s}{N}.$$

Example. *The binomial random variable with parameters n and p.* Let X be the number of successes in n independent trials of an experiment, if each trial is successful with probability p. We know from previous work that:

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}, \quad x = 0, 1, 2, \ldots n.$$

We can calculate the expectation as follows:

$$\begin{aligned}
E(X) &= \sum_{x=0}^{k} x \binom{n}{x} p^x \, (1-p)^{n-x} \\
&= \sum_{x=1}^{k} x \binom{n}{x} p^x \, (1-p)^{n-x} \\
&= \sum_{x=1}^{k} n \binom{n-1}{x-1} p \, p^{x-1} \, (1-p)^{(n-1)-(x-1)} \\
&= n \, p \sum_{x=1}^{k} \binom{n-1}{x-1} p^{x-1} \, (1-p)^{(n-1)-(x-1)} \\
&= n \, p \sum_{x=0}^{k-1} \binom{n-1}{x} p^{x} \, (1-p)^{(n-1)-(x)} \\
&= n \, p
\end{aligned}$$

Example. *The geometric random variable with parameter p.* Suppose a trial is successful with probability p. Suppose that independent trials are repeated over and over. Let X be the number of independent trials required to obtain the first success. We know from previous work that:

$$P(X = i) = (1 - p)^{i-1} p, \quad i = 1, 2, \ldots.$$

We can calculate the expectation as follows:

$$\begin{aligned}
EX &= \sum_{i=1}^{\infty} i(1-p)^{i-1} p \\
&= p \sum_{i=1}^{\infty} i(1-p)^{i-1}.
\end{aligned}$$

Now, if $0 \leq x < 1$, then $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$, so:

$$\begin{aligned}\sum_{i=1}^{\infty} i x^{i-1} &= \frac{d}{dx} \sum_{i=0}^{\infty} x^i = \frac{d}{dx} \frac{1}{1-x} \\ &= (-1)(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}\end{aligned}$$

Applying this when $x = 1 - p$, we get:

$$E(X) = p \, \frac{1}{(1-(1-p))^2} = \frac{1}{p}.$$