Lecture 22. Change of variables

See Chapter 8, section 4.

Suppose H is a function from a domain the x-y-plane to the u-v-plane. We can express the dependence of (u,v) on (x,y) by writing:

$$(u,v) = (u(x,y), v(x,y)) = H(x,y).$$

Here, $u = u(x,y)$ and $v = v(x,y)$ are two real-valued functions, each of two variables.

We say H is invertible if there is a function G defined on a domain in the u-v-plane such that:

$$G(H(x,y)) = (x,y) \qquad \text{and} \qquad H(G(u,v)) = (u,v).$$

If there is such a function, we can view x and y as functions of u and v:

$$(x,y) = (x(u,v), y(u,v)) = G(u,v).$$

Examples.

1. Suppose $H(x,y) = (x+y, x-y)$, so $u = x+y$, $v = x-y$. Then the inverse of H is $G(u,v) = (\frac{u+v}{2}, \frac{u-v}{2})$, so $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$.
2. Suppose $H(x,y) = (x+y, \frac{y}{x})$, so

$$u = x+y, \quad v = \frac{y}{x} \qquad (x \neq 0).$$

We can solve for x and y by noting that $y = xv$, so $u = x + xv = x(1+v)$, so

$$x = \frac{u}{1+v}, \quad y = \frac{u\,v}{1+v} \qquad (u \neq 0\,, v \neq -1).$$

Thus, if we restrict H to the portion of the x-y-plane where $x \neq 0$ and $y \neq -x$, then it has an inverse $G(u,v) = (\frac{u}{1+v}, \frac{u\,v}{1+v})$, defined on the set where $u \neq 0$ and $v \neq -1$.

Suppose H and G are as above. Let

$$G'(u,v) := \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix},$$

and let

$$J(u,v) := \det G'(u,v) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Suppose A is contained in the domain of H in the x-y-plane and

$$B := \{\, H(x,y) \mid (x,y) \in A \,\} = H(A) \subseteq \text{ the } u\text{-}v\text{-plane}.$$

The change of variables theorem (which you learned in Math 2057) says: [1]

$$\int\int_A f(x,y)\,dy\,dx = \int\int_B f(x(u,v),y(u,v))\,|J(u,v)|\,du\,dv. \tag{1}$$

Now if $f = f_{X,Y}$ is the pdf of (X,Y), the left side of (1) is $P((X,Y)\in A) = P(H(X,Y)\in H(A)) = P((U,V)\in B)$. But then

$$P((U,V)\in B) = \int\int_B f(x(u,v),y(u,v))\,|J(u,v)|\,du\,dv.$$

Since this us true for all B, the joint pdf of (U,V) is

$$f_{U,V}(u,v) = f_{X,Y}(x(u,v),y(u,v))\mid J(u,v)\mid. \tag{2}$$

On the following pages, we will work some problems.

[1] Some people may prefer the following way of phrasing the change of variables theorem:

$$\int\int_{G(B)} f(x,y)\,dy\,dx = \int\int_B f\big(G(u,v)\big)\,|\det G'(u,v)|\,du\,dv. \tag{1'}$$

This way of stating it makes obvious the similarity with the one-dimensional case:

$$\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f\big(g(u)\big)\,g'(u)\,du.$$

Here, the change of variables (or substitution) $x = g(u)$ has simplified the right hand side. There is no absolute value in the right hand side because the order of the limits of integration keep track of the orientation. For example, if g is decreasing on $[a,b]$, g' is negative but there is also a factor of -1 on the left side due to the fact that $g(a) > g(b)$:

$$\int_{g([a,b])} f(x)\,dx = -\int_{g(a)}^{g(b)} f(x)\,dx = -\int_a^b f\big(g(u)\big)\,g'(u)\,du = \int_{[a,b]} f\big(g(u)\big)\,|g'(u)|\,du.$$

Problem 1. Suppose X and Y are random variables with joint pdf $f_{X,Y}$. Let $U = X+Y$ and $V = X - Y$. Find the joint pdf $f_{U,V}$.

Solution. We know from the examples given earlier that $X = \frac{U+V}{2}$, and $Y = \frac{U-V}{2}$. Hence $J(u,v) = (\frac{1}{2})(\frac{-1}{2}) - (\frac{1}{2})(\frac{1}{2}) = \frac{-1}{2}$. Thus, by Equation (2),

$$f_{U,V}(u,v) = \frac{f_{X,Y}(\frac{u+v}{2}, \frac{u-v}{2})}{2}. \qquad /////$$

Problem 2. Suppose X and Y are random variables with joint pdf $f_{X,Y}$. Let $U = X+Y$ and $V = Y/X$. Find the joint pdf $f_{U,V}$.

Solution. We know from the examples given earlier that $X = \frac{U}{V+1}$, and $Y = \frac{U\,V}{V+1}$. Now

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{1}{1+v} & \frac{-u}{(1+v)^2} \\ \frac{v}{1+v} & \frac{u}{(1+v)^2} \end{pmatrix}.$$

Hence

$$J(u,v) = \frac{1}{1+v}\,\frac{u}{(1+v)^2} - \frac{-u}{(1+v)^2}\,\frac{v}{1+v} = \frac{u}{(1+v)^2}.$$

Thus, by Equation (2),

$$f_{U,V}(u,v) = f_{X,Y}\left(\frac{u}{1+v}, \frac{u\,v}{1+v}\right)\frac{|u|}{(1+v)^2}. \qquad /////$$

One of the most common applications of Equation (2) *is calculating the pdf of a function of two random variables, given the joint pdf of the two. The strategy is to reduce the problem to the task of calculating a marginal pdf.*

Problem 3. Suppose X and Y are random variables with joint pdf $f_{X,Y}$. Find the pdf of $X+Y$ in terms of $f_{X,Y}$.

Solution. Let $U = X+Y$ and let $V = Y$. Then $X = U - V$, $Y = V$ and

$$J(u,v) = (1)\,(1) - (-1)\,(0) = 1.$$

Thus, by Equation (2),

$$f_{U,V}(u,v) = f_{X,Y}\big(u-v, v\big).$$

To find f_U, we integrate with respect to v:

$$f_U(u) = \int_{-\infty}^{\infty} f_{X,Y}\big(u-v, v\big)\,dv.$$

If X and Y are independent, then $f_{X,Y}(x,y) = f_X(x)f_Y(y)$, so

$$f_U(u) = \int_{-\infty}^{\infty} f_X(u-v)f_Y(v)\,dv.$$

Problem 4. Suppose X and Y are random variables with joint pdf $f_{X,Y}$. Find the pdf of XY in terms of $f_{X,Y}$.

Solution. Let $U = XY$ and let $V = Y$. Then

$$X = U/V\ ,\ Y = V.$$

Now

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ 0 & 1 \end{pmatrix}.$$

Accordingly, $J(u,v) = (\frac{1}{v})\,(1) - (\frac{-u}{v^2})\,(0) = \frac{1}{v}$. Thus, by Equation (2),

$$f_{U,V}(u,v) = f_{X,Y}\big(u/v, v\big)\,\frac{1}{|v|}.$$

To find f_U, we integrate with respect to v:

$$f_U(u) = \int_{-\infty}^{\infty} \frac{f_{X,Y}\big(u/v, v\big)}{|v|}\,dv.$$

If X and Y are independent, then $f_{X,Y}(x,y) = f_X(x) f_Y(y)$, so

$$f_U(u) = \int_{-\infty}^{\infty} f_X(u/v)\,f_Y(v)\,\frac{1}{|v|}\,dv.$$

Example. One of the most important changes of variables is the linear substitution:

$$\begin{pmatrix} u \\ v \end{pmatrix} = H(x,y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix},$$

which has inverse:

$$\begin{pmatrix} x \\ y \end{pmatrix} = G(u,v) = (ad-bc)^{-1}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = (ad-bc)^{-1}\begin{pmatrix} du-bv \\ -cu+ay \end{pmatrix}.$$

In this case, $J(u,v) = (ad-bc)^{-1}$, and

$$f_{U,V}(u,v) = f_{X,Y}\big(G(u,v)\big)(ad-bc)^{-1}.$$