Lecture 23.

Some topics from 10.1

Fact. Expectation is linear: $\mathrm{E}(aX) = a\mathrm{E}(X)$ for any constant a and any random variable X and $\mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(Y)$ for any random variables X and Y. Thus, for any constants a_i and random variables X_i,

$$\mathrm{E}(a_1X_1 + \ldots + a_nX_n) = a_1\mathrm{E}(X_1) + \ldots + a_n\mathrm{E}(X_n).$$

Comment. Of course, for the statements in the Fact to be a meaningful, it is necessary that $\mathrm{E}(X)$, $\mathrm{E}(Y)$ and all the $\mathrm{E}(X_i)$ exist. In the following, we will deal only with random variables that have and expected value.)

Question. Give an example of a discrete random variable that has no expected value.

Example. If a die is rolled n times, the expected sum of the rolls is n time the expected outcome of a single roll. More generally, if any experiment with numerical outcomes is repeated n times, the expected sum of the outcomes is n times the expected outcome of a single trial.

Example 10.3. If m deaths occur randomly among n couples, what is the expected number of intact couples? To determine this we use a useful trick: let X_i be the random variable that has value 1 if the i^{th} couple is intact and has value 0 otherwise. Then to solve our problem, it suffices to find $\mathrm{E}(X_i) = P(X_i = 1)$, the probability that the i^{th} couple is intact. (Why?) This is the same for all i, and is equal to the probability that all m deaths occur among the $2n-2$ people other than the i^{th} couple. Thus, $P(X_i = 1) =$

$$\frac{\binom{2n-2}{m}}{\binom{2n}{m}} = \frac{((2n-2)-(m-2))(2n-2)-(m-1))}{2n(2n-1)} = \frac{(2n-m)(2n-m-1))}{2n(2n-1)}.$$

The answer is n times this number. (Why?) (How might this problem be generalized?)

Exploration. (Compare 10.4 and 10.5). There are s special items among n items in a box. If items are removed one at a time without replacement, what is the expected number of items that need to be taken to collect all the special items? *Hints toward a solution.* Let S be the collection of all s-element subsets of $\{1, 2, \ldots, n\}$. For each $\sigma \in S$, let E_σ be the event that special items are taken on each of the draws indexed by σ (and of course non-special items are taken on all other draws). Let X_σ be the random variable that equals $\max(\sigma)$ if E_σ is 0 otherwise. Then the number of draws required is $\sum_{\sigma\in S} X_\sigma$. Also, $\mathrm{E}(X_\sigma) = \max(\sigma)\cdot P(E_\sigma)$.

Homework. Problems 1–5 in 10.1.