Lecture 3. Combinatorial Constructions

Many probability spaces arise from combinatorial structures.

Permutations. If a numbered set of n items is rearranged in a new order (e.g., 123456 might be rearranged as 352461), there are n possibilities for the first element, $n-1$ for the second, $n-2$ for the third, etc. Therefore the total number of possible rearrangements is

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1.$$

Fact 1. *There are $n!$ ways of arranging n distinct elements in order.*

Consider an experiment consists in rearranging an n-element set randomly in such a way that all rearrangements are equally probable. We can model this with a probability space in which the $n!$ rearrangements are the outcomes and each has an assigned probability mass of $1/n!$. For example, after many shuffles of a deck of cards there is no rearrangement that is more probable than any other. Thus, each card sequence has a probability of $1/52!$. ($52!$ is about 8×10^{67}.)

If k elements are taken from an n-element set in order, without replacement, there are n possibilities for the first choice, $n-1$ for the second, $n-2$ for the third, ... and $n-k+1$ for the k^{th} choice. Therefore, the total number of ways of completing this task is

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-k+1) = \frac{n!}{(n-k)!}.$$

Fact 2. *There are $\frac{n!}{(n-k)!}$ ways of selecting k elements in order from a collection of n distinct elements.*

For example, there are $52 \cdot 51 \cdot 50 = 132,600$ ways to be dealt a hand of three cards. (Note that there are NOT $132,600$ three-card hands, since each 3-card hand has $3! = 6$ ways of being ordered and hence 6 ways of being dealt in order. This foreshadows what we will say about combinations, below.)

Problem. What is the probability that a three-card hand contains an ace?

Solution. We work in the probability space of ordered three-card hands. We need to add up the probability masses of the outcomes in the event consisting of all hands that include an ace. Since all the hands have the same probability, we can find this probability by counting the number of outcomes in this event and then multiplying by $1/132,600$, this being the probability of a hand. There are several ways to make the count. a) There are $51 \cdot 50$ hands that have the ace of spades as a first card. The same is true for the other three suits, so there $4 \cdot 51 \cdot 50$ hands that have an ace first. There are $48 \cdot 51 \cdot 50$ cards that don't have an ace first, and of these, there are $48 \cdot 4 \cdot 50$ that get their first ace on the second deal. Finally, there are $48 \cdot 47 \cdot 4$ hands that get their first ace on the on the third deal. Now, $4 \cdot 51 \cdot 50 + 48 \cdot 4 \cdot 50 + 48 \cdot 47 \cdot 4 = 28,824$. b) There is a much more elegant way to count. The number of ordered hands that **contain no ace** is $48 \cdot 47 \cdot 46 = 103,776$.

Therefore, the number of hands **with** and ace is $132,600 - 103,776 = 28,824$. Either way it's figured, the probability of the even is

$$\frac{28,824}{132,600} = 0.217\ldots.$$

Remark. The solution illustrates a valuable strategy: if X is a finite set of known size, and A is a subset, then we may determine the number of elements in A by counting directly, or by counting the number of elements of X that are *not* in A. Sometimes the latter is much easier than the former. Similarly, if A is an event, then it may be much easier to compute the probability that A does *not* occur than to compute the directly the probability that it does. But the former determines the latter.

Problem. The letters of "$MISSISSIPPI$" are scrambled. What is the probability that they still spell $MISSISSIPPI$?

Solution. There are 11 letters in the word, so there are 11! ways of scrambling them, each with probability 1/11!. However, because some letters are repeated, some outcomes spell the same word. To make this clear, we label the repeated letters:

$$MI_1S_1S_2I_2S_3S_4I_3P_1P_2I_4.$$

Then, a scrambling such as the following preserves the word:

$$MI_4S_2S_3I_1S_4S_1I_2P_2P_1I_3.$$

There are 4! ways to scramble the Ss, 4! ways to scramble the Is and 2! ways to scramble the Ps, making a total of $4! \cdot 4! \cdot 2!$ scramblings that leave the word unchanged. Thus, the probability that the letters still spell $MISSISSIPPI$ after scrambling is:

$$\frac{4! \cdot 4! \cdot 2!}{11!} = \frac{1}{34650}.$$

Combinations. Above, we found that there are $n!/k!$ ways of selecting k elements from and n-element set, *provided that we count two selections as different when the order is different*. If we are only interested when the same items are selected with the order ignored, then we need a different formula.

Fact 3. *A set with* n *elements has* $\frac{n!}{k!(n-k)!}$ *different subsets of size* k*.*

Proof. This is a consequence of Facts 1 and 2. By Fact 2, there are $\frac{n!}{(n-k)!}$ ways of choosing k elements in order. By Fact 1, there are $k!$ ways of ordering k elements. That is to say, the same k elements may be chosen-in-order $k!$ different ways. /////

Example. Within each line in the array below, you see the same 3 letters from $\{a, b, c, d, e\}$, arranged in each of the six possible ways. There are 60 ways of choosing 3 letters from

$\{a, b, c, d, e\}$ in order. But there are only 10 distinct subsets of $\{a, b, c, d, e\}$.

abc	acb	bac	bca	cab	cba
abd	adb	bad	bda	dab	dba
abe	aeb	bae	bea	eab	eba
acd	adc	cad	cda	dac	dca
ace	aec	cae	cea	eac	eca
ade	aed	dae	dea	ead	eda
bcd	bcd	cbd	cdb	dbc	dcb
bce	bce	cbe	ceb	ebc	ecb
bde	bde	dbe	deb	ebd	edb
cde	cde	dce	dec	ecd	edc

Notation. $\binom{n}{k}$ (pronounced "n choose k) is the following number:

$$\binom{n}{k} := \frac{n!}{k!(n-k)!} = \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdots \frac{n-k+1}{k}.$$

$\binom{n}{k}$ is the number of k-element subsets of and n-element set.

Problem. There are r red beads and g green beads in a jar. We remove k beads. What is the probability that all k are the same color?

Solution. There are $\binom{r+g}{k}$ subsets of the set of all beads. Each is equally probable, so the mass assigned to each is $\binom{r+g}{k}^{-1}$ There are $\binom{r}{k}$ sets that are all red and $\binom{g}{k}$ that are all green, so the number of outcomes in the event we are interested in is $\binom{r}{k} + \binom{g}{k}$. Thus, the probablity of the event is:

$$\frac{\binom{r}{k} + \binom{g}{k}}{\binom{r+g}{k}} = \frac{\frac{r!}{k!(r-k)!} + \frac{g!}{k!(g-k)!}}{\frac{(r+g)!}{k!(r+g-k)!}} = \frac{r(r-1)\cdots(r-k+1) + g(g-1)\cdots(g-k+1)}{(r+g)(r+g-1)\cdots(r+g-k+1)}$$

Assignement. TBA