Lecture 4. Examples

Problem 1.1. A ship is carrying r Russians and g Georgians and no one else. A committee of k people is to be selected randomly to assume the task of rationing the rum. What is the probability that there will be x Russians on the rum committee?

Solution. The sample space (set of all outcomes) is the set of all committees. There are $\binom{r+g}{k}$ elements, each equally likely. We must count the number of committees with x Russians. There are $\binom{r}{x}$ ways to pick the Russians, but then we must also pick $n-k$ Georgians, and there are $\binom{g}{k-x}$ way to do this. Thus, there are $\binom{r}{x} \cdot \binom{g}{k-x}$ ways to make a committee with x Russians, so the desired probability is:

$$\frac{\binom{r}{x} \cdot \binom{g}{k-x}}{\binom{r+g}{k}.}$$

Problem 1.2. The last problem in Lecture 3 had

$$\frac{\binom{r}{k} + \binom{g}{k}}{\binom{r+g}{k}}$$

as its answer. Why do both problems have the same denominator? Why does one problem have addition in the numerator, while the other has multiplication?

Problem 1.3. At a tea party, a lady claimed to be able to tell by taste alone whether, in a cup of tea with milk, the tea or the milk had been poured first. The famous statistician R.A.Fisher was present, and he proposed presenting the lady with 8 cups, four of each kind, in random order and asking her to classify them. She was told that there were four of each kind, so her task was essentially to identify the four milk-first cups. It is said that she accomplished the task. What would the chances of that be, if the lady were merely guessing?

Problem 1.4. Suppose that among the registered voters in a city there are 3 party-members for every Independent. Suppose there are N Independents, and hence $4N$ voters all together. If 100 voters are chosen at random, then in terms of N what is the probability that 25 of them are Independents? What is the probability that the number of independents in the sample is between 20 and 30 (inclusive)? Use Mathematica to find the probabilities when $N = 25$, $N = 100$, $N = 1000$, $N = 10000$, $N = 1000000$.

Problem 1.5. An urn contains n balls, and s of them are special. A random set of k balls is taken out. What is the probability that x of those taken are special?

Problem 1.6. In what respect are problems 1.1–1.5 similar to one another?

Homework Project. An urn contains 20 black balls and 20 white balls. The balls are placed randomly in 20 cups, two balls to a cup. What is the probability that:

A) every cup contains a black-white pair?

B) no cup contains a black-white pair?

C) exactly k cups contain black-white pairs?

Solution to parts A) and B). The sample space for this experiment consists of all ways of placing the 40 balls into the 20 cups so that each cup contains two. There are $\binom{40}{2}$ ways of filling the first cup, $\binom{38}{2}$ ways to fill the second, and so on. This means that there are

$$\left(\frac{40}{1}\cdot\frac{39}{2}\right)\left(\frac{38}{1}\cdot\frac{37}{2}\right)\cdots\left(\frac{2}{1}\cdot\frac{1}{2}\right)=\frac{40!}{2^{20}}$$

ways to fill the cups, each equally likely.

Now, we must count the outcomes in the events that we are considering:

A) The event that every cup contains a black-white pair. There are 20! ways of placing the 20 black balls in the 20 cups, and there is the same number of ways of placing the white balls. Thus, there are $20!\cdot 20!$ outcomes in this event. The probability is:

$$\frac{20!\cdot 20!\cdot 2^{20}}{40!}\approx 7.60684\times 10^{-6}.$$

B) The event that no cup contains a black-white pair. To describe an outcome in this event, we first select 10 cups to hold the 10 pairs of black balls; there are $\binom{20}{10}$ ways to do this. Then we place the black balls in these cups; there are $20!/2^{10}$ ways. Finally, we place the white balls in the remaining cups, and there are $20!/2^{10}$ ways to do this. Accordingly, there are $\binom{20}{10}\cdot\frac{20!}{2^{10}}\cdot\frac{20!}{2^{10}}$ outcomes in this event, and the probability is:

$$\binom{20}{10}\cdot\frac{20!\cdot 20!}{40!}\approx 1.3403\times 10^{-6}.$$

Problem. Using the kind of reasoning applied in parts A) and B), find the probability of the event that exactly k cups contain black-white pairs. **Hints:** There will be k black balls in the cups containing black-white pairs, and the remaining black balls will be in pairs. Thus, k must be even for the probability of the event to be non-zero, and it may be any even number between 0 and 20, inclusive. Let us suppose $k=2n$, where n is an integer between 0 and 10. Now, compute the following:

a) the number of ways of choosing $2n$ cups from 20,

b) the number of ways of choosing $2n$ black balls from 20,

c) the number of ways of choosing $2n$ white balls from 20,

d) the number of ways of placing the $2n$ black balls in the $2n$ cups, one ball to a cup,

e) the number of ways of placing the $2n$ white balls in the $2n$ cups, one ball to a cup,

f) the number of ways of choosing $10-n$ cups (to hold the black balls) from the $20-2n$ remaining, *g*) the number of ways of putting the remaining $20 - 2n$ black balls in these cups, two balls to a cup,

h) the number of ways of putting the remaining $20 - 2n$ white balls in the remaining $10 - n$ cups, two balls to a cup.

If we multiply the numbers $(a), \ldots, (h)$ together, we get the number of outcomes in the event. Do this in Mathematica.