Lecture 5. Conditional Probability and Independence

Definition. Suppose (Ω, P) is a probability space and A is an event such that $P(A) \neq 0$. Let B be another event. We define the *conditional probability of B given A*—denoted $P(B|A)$—by:

$$P(B|A) := \frac{P(A \cap B)}{P(A)}.$$

Example. Ted and Alice have two children. Given that they have a boy, what is the probability that they have two boys?

Solution. With no knowledge of the genders of their children, we would assume that gg, gb, bg and bb are equally probable, since there are equal chances of having a girl or a boy as first child, and the same is true for the second regardless of the gender of the first. But since we know they have a boy, only gb, bg and bb are possible. These being equally probable, the answer must be 1/3. /////

Comment. To connect the example to the definition, we name the events that come into play. Call the event of having a boy B. Then $B = \{gb, bg, bb\}$. Call the event of having two boys A. Then $A = \{bb\}$, and $A \cap B = \{bb\}$.

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

Comment. This problem seems to violate the requirement that "an experiment is a repeatable action." One might argue that Ted and Alice either have two boys or they don't, so the probability they have two boys is either 1 or 0, and we just don't know which. In questions of this type, the reader is implicitly asked to view Ted and Alice as a random couple. A different perspective is sometimes taken, namely the "subjectivist interpretation of probability." In this view, probability statements are statements of quantified certainty—as when one says, "I'm 60% certain that I will get a B on the next quiz." The same rules apply, however, regardless of the interpretation.

Example. Two fair coins are flipped. Given that at least one head is obtained, what is the probability that two heads are flipped?

Solution. This is exactly the same as the Ted and Alice problem. /////

Example. Two dice are rolled. Given that at least one of the dice has come up on an odd number other than 1, what is the probability that the sum is 7?

Solution. In the table below, the rows are labelled by the outcome i of the first die and the columns by the outcome j of the second. Each of the 36 values for (i, j) is equally probable.

	1	2	3	4	5	6
1	$(1,1)$	$(1,2)$	$(\mathbf{1},\mathbf{3})$	$(1,4)$	$(\mathbf{1},\mathbf{5})$	$(1,6)$
2	$(2,1)$	$(2,2)$	$(\mathbf{2},\mathbf{3})$	$(2,4)$	$(\mathbf{2},\mathbf{5})$	$(2,6)$
3	$(\mathbf{3},\mathbf{1})$	$(\mathbf{3},\mathbf{2})$	$(\mathbf{3},\mathbf{3})$	$(\mathbf{3},\mathbf{4})$	$(\mathbf{3},\mathbf{5})$	$(\mathbf{3},\mathbf{6})$
4	$(4,1)$	$(4,2)$	$(\mathbf{4},\mathbf{3})$	$(4,4)$	$(\mathbf{4},\mathbf{5})$	$(4,6)$
5	$(\mathbf{5},\mathbf{1})$	$(\mathbf{5},\mathbf{2})$	$(\mathbf{5},\mathbf{3})$	$(\mathbf{5},\mathbf{4})$	$(\mathbf{5},\mathbf{5})$	$(\mathbf{5},\mathbf{6})$
6	$(6,1)$	$(6,2)$	$(\mathbf{6},\mathbf{3})$	$(6,4)$	$(\mathbf{6},\mathbf{5})$	$(6,6)$

Let $A := \{ (i,j) \in \Omega \mid \text{either } i \text{ is an odd number other than 1 or } j \text{ is} \}$. The entries in the table that belong to A are in bold face. There are 20 of them, so $P(A) = \frac{20}{36} = \frac{5}{9}$. Now, let $B := \{ (i,j) \in \Omega \mid i + j = 7 \}$. Then $A \cap B$ has four outcomes of the 36 in Ω, so $P(A \cap B) = \frac{4}{36} = \frac{1}{9}$. Thus,

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/9}{5/9} = \frac{1}{5}. \qquad /////$$

Exercise 1. A coin is flipped 5 times. What is the probability that at least 2 tails are obtained, given that at least 2 heads are obtained? (Comment. You may think of this problem in the following way. Suppose the experiment of flipping a fair coin 5 times is repeated over and over again, but we only record the outcome if at least 2 heads are obtained. In what proportion of the recorded outcomes will we find at least two tails?)

Solution. The sample space is $\{ HHHHH, HHHHT, HHHTH, HHHTT, \dots \}$, and it includes 32 outcomes, each with probability mass 1/32. Let C be the event of getting at least 2 heads, and let D be the event of getting at least two tails. We want to compute $P(D|C) = \frac{P(C \cap D)}{P(C)}$. Now, $C \cap D$ is the event of getting at least two heads *and* at least two tails. This is the same as the event of getting either two or three heads. There are $\binom{5}{2} = 10$ ways of getting two heads, and $\binom{5}{3} = 10$ ways of getting three heads, so the number of outcomes in $C \cap D$ is 20, and $P(C \cap D) = 20/32$. C is the event of getting 2, 3, 4 or 5 heads, and there are $\binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 10 + 10 + 5 + 1$ outcomes in C. Thus $P(C) = 26/32$. It follows that $P(D|C) = 20/26 = 10/13$. /////

Exercise 2. Suppose that A and B are events such that $P(A \cap B) \neq 0$. Let C be another event. Show that

$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B).$$

Exercise 3. Suppose $P(A_1) \neq 0$ and $P(A_2) \neq 0$, $A \subseteq A_1 \cup A_2$ and $A \cap A_1 \cap A_2 = \emptyset$. Show that

$$P(A) = P(A_1)P(A|A_1) + P(A_2)P(A|A_2).$$

Definition. Suppose (Ω, P) is a probability space and A and B are events. We say *A and B are independent* if

$$P(A \cap B) = P(A)P(B).$$

Exercise 4. Are mutually exclusive events independent? If A and B are independent and B and C are independent, does it necessarily follow that A and C are independent?

Two-by-two Tables

If A and B are events in a probability space (Ω, P), then there is a four-way classification of outcomes:

$$\begin{array}{cc} \omega \in A \text{ and } \omega \in B & \omega \in A \text{ and } \omega \notin B \\ \omega \notin A \text{ and } \omega \in B & \omega \notin A \text{ and } \omega \notin B, \end{array}$$

corresponding to the four sets

$$\begin{array}{cc} A \cap B & A \cap B^c \\ A^c \cap B & A^c \cap B^c. \end{array}$$

We may put the probabilities of the events in a table, as follows:

	B	B^c	row sum
A	$P(A \cap B)$	$P(A \cap B^c)$	$P(A)$
A^c	$P(A^c \cap B)$	$P(A^c \cap B^c)$	$P(A^c)$
column sum	$P(B)$	$P(B^c)$	1

The probabilities $P(A)$ and $P(A^c)$ are called the *(row) marginals* and $P(B)$ and $P(B^c)$ are called the *(column) marginals*.

If we abbreviate $x := P(A \cap B)$, $y := P(A \cap B^c)$, $z := P(A^c \cap B)$, $w := P(A^c \cap B^c)$, then $w = 1 - x - y - z$. The entries in the table are as follows:

	B	B^c	row sum
A	x	y	$x + y$
A^c	z	$1 - x - y - z$	$1 - x - y$
column sum	$x + z$	$1 - x - z$	1

The row marginals are obtained by summing the rows, and the column marginals are obtained by summing the columns. Note that there are nine numerical entries (including the 1 in the lower right) in the table, but there are dependencies among them. We can determine all the entries from any three main entries or from certain combinations of marginal and main entries.

Conditional probabilities are related to the entries in the table in the following way:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{x}{x+y} \qquad \text{and} \qquad P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{x}{x+z}.$$

Exercise 5. Complete the table:

	B	B^c	row sum
A			0.3
A^c		0.1	
column sum	0.7		

Exercise 6. In a certain population, 40% are infected, 30% are symptomatic and 50% are neither infected nor symptomatic. What is the probability that a random individual is infected but not symptomatic? Symptomatic but not infected. Both symptomatic and infected? Given that an individual is symptomatic, what is the probability that he is infected?