Lecture 6. Bayes's Formula

Review

1. If A is an event such that $P(A) \neq 0$ and B be another event, then the *conditional probability of B given A* is:
$$P(B|A) = \frac{P(A \cap B)}{P(A)}. \tag{1}$$

2. We say *A and B are independent* if
$$P(A \cap B) = P(A)P(B).$$

3. If A and B are events, we may put the probabilities of the events definable from A and B in a table, as follows:

	B	B^c	$+$
A	$x = P(A \cap B)$	$y = P(A \cap B^c)$	$x + y = P(A)$
A^c	$z = P(A^c \cap B)$	$1 - x - y - z = P(A^c \cap B^c)$	$1 - x - y = P(A^c)$
$+$	$x + z = P(B)$	$1 - x - z = P(B^c)$	1

Then $P(B|A) = \frac{x}{x+y}$. If $P(B) \neq 0$, then $P(A|B) = \frac{x}{x+z}$.

Bayes's Formula

If A and B both have non-zero probability, then equation (1) tells us:
$$P(A|B)P(B) = P(A \cap B) = P(B|A)P(A).$$

From this, we get **Bayes's Formula (simple form):**
$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}. \tag{2}$$

Suppose $A_1, A_2, \cdots, A_n$ are disjoint and $B \subseteq A_1 \cup A_2 \cup \cdots \cup A_n$. We have the following

Decomposition formula:
$$\begin{aligned} P(B) &= P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B) \\ &= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_n)P(A_n). \end{aligned} \tag{3}$$

From (2) and (3) we get

Bayes's Formula: If $A_1, A_2, \cdots, A_n$ are disjoint sets and $B \subseteq A_1 \cup A_2 \cup \cdots \cup A_n$, then for any $k \in \{1, 2, \ldots, n\}$:
$$P(A_k|B) = \frac{P(B|A_k)P(A_k)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}.$$

Comment. The simple form of Bayes's formula is usually adequate, if you are prepared to think a bit. The full form can often be applied directly (and more thoughtlessly). Here is an example that illustrates this.

Example/illustration

A child has run away. It is known that she must be in one of three locations, and the probability of being in region i is estimated to be α_i. (Note that $\alpha_1 + \alpha_2 + \alpha_3 = 1$.) It is also estimated that *if the child is in location i, then a search of that location will* *<u>fail</u>* *to find her* with probability β_i, $i = 1, 2, 3$. If a search of region 1 has failed to find her, what is the probability that she is in that region anyway?

Solution 1. Let A_i be the event that she is in region i. Let B be the event that a search of region 1 is unsuccessful. We are asked to find $P(A_1|B)$. Now,

$$\begin{aligned}
P(A_1|B) &= \frac{P(B|A_1)P(A_1)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3)} \\
&= \frac{\beta_1\alpha_1}{\beta_1\alpha_1 + \alpha_2 + \alpha_3} \\
&= \frac{\alpha_1\beta_1}{\alpha_1\beta_1 + (1-\alpha_1)}.
\end{aligned}$$

Solution 2. We can use a 2-way table to solve this, as follows. Let A_1 be the event that she is in region 1, and let B be the event that a search of region 1 is unsuccessful. Then

	B	B^c	$+$
A_1	$\alpha_1\beta_1$	?	α_1
A_1^c	?	0	?
$+$	?	?	1

The explanations for the entries are as follows:

- $\alpha_1 = P(A_1)$ by assumption.
- $\beta_1 = P(B|A_1) = P(B \cap A_1)/P(A_1)$ by definition of β_1, so $\alpha_1\beta_1 = P(A_1 \cap B)$.
- B^c is the event that a search of region 1 is successful, but this cannot happen if the child is not in region 1. This accounts for the 0.

Now, the other boxes are easily filled in:

	B	B^c	$+$
A_1	$\alpha_1\beta_1$	$\alpha_1(1-\beta_1)$	α_1
A_1^c	$1-\alpha_1$	0	$1-\alpha_1$
$+$	$\alpha_1\beta_1 + (1-\alpha_1)$	$\alpha_1(1-\beta_1)$	1

Hence,

$$P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{\alpha_1\beta_1}{\alpha_1\beta_1 + (1-\alpha_1)}.$$

Homework Problem. If regions 1 and 2 have been searched without success, what is the probability that the child is in region 3?

Seeing independence in two-by-two tables

We have seen that three of the entries in a two-by-two table may suffice to determine all of them, but even when we choose three positions that suffice, we are not free to fill them in any way we choose. For example, $P(A)$, $P(B)$ and $P(A \cap B)$ tosuffice determine all the entries, but $P(A \cap B)$ must be less that or equal to the minimum of $P(A)$ and $P(B)$. In this case, this is the only restriction, so if we have $0 \leq x \leq 1$ and $0 \leq y \leq 1$ in the following table, and $0 \leq z \leq \min\{x, y\}$, then the rest of the entries are between 0 and 1. (Note that we are using x, y and z to denote different entries than we did previously.)

	B	B^c	$+$
A	z	$x - z$	x
A^c	$y - z$	$1 - x - y + z$	$1 - x$
$+$	y	$1 - y$	1

That the triple (x, y, z) can be at any position in the part of the unit cube illustrated in Notebook 6, so each point in this pyramid corresponds to a table. A and B are independent if and only if $z = xy$. Thus, the tables corresponding to independent events lie on the surface defined by this equation. This is also illustrated in Notebook 6.

Homework 3.2: 9. What is the probability that no hearts are dealt before the ace of spades?

There are two ways to approach this problem.

Solution 1. For the sample space, we take the set of all 52! ways of dealing 52 cards. Let E be the event that the ace of spades occurs before any hearts. Then E is the disjoint union of the event E_k, where

$$E_k := \{\text{exactly } k \text{ non-hearts have been dealt when the ace of spades is dealt}\}.$$

Now,

$$\begin{aligned}
\#(E_0) &= 51! \\
\#(E_1) &= 38 \cdot 50! \\
\#(E_2) &= 38 \cdot 37 \cdot 49! \\
&\vdots \\
\#(E_k) &= 38 \cdot 37 \cdots (39 - k)(51 - k)! \\
\#(E_{38}) &= 38 \cdot 37 \cdots (1) \cdot 13! \\
\#(E_{39}) &= 0
\end{aligned}$$

Thus,

$$\#(E) = \sum_{k=0}^{38} \binom{38}{k} k!(51 - k)!,$$

and

$$P(E) = \sum_{k=0}^{38} \binom{38}{k} k!(51-k)!/52! = \frac{1}{14}.$$

Solution 2. Let the sample space consist of all ways of dealing 52 cards. Now, we can think of each element of this space as consisting of 14 designated cards—namely the 13 hearts and the ace of spades—laid down in some order, with the remaining cards interspersed in some manner. Regardless of the order of the 14, there are the same number of ways of interspersing the remaining cards, so we have a partition of sample space into 14! events, each of equal probability. This shows that we may, in fact, discard the original sample space, and use work the problem in terms of a sample space consisting of the 14! ways of ordering the hearts and the ace of spades. Now the ace comes in the first position in exactly 1/14 of the elements of this space. This is an elegant solution, and it shows many other things about the experiment as well. For example, the probability that the ace of spades is dealt after exactly 3 hearts is 1/14.