EXERCISES 3.5

A

1. Jean le Rond d'Alembert, a French mathematician, believed that in successive flips of a fair coin, after a long run of heads, a tail is more likely. Do you agree with d'Alembert on this? Explain.

2. Clark and Anthony are two old friends. Let A be the event that Clark will attend Anthony's funeral. Let B be the event that Anthony will attend Clark's funeral. Are A and B independent? Why or why not?

3. In a certain country, the probability that a fighter plane returns from a mission without mishap is 49/50, *independent* of other missions. In a conversation, Mia concluded that any pilot who flew 49 consecutive missions without mishap should be returned home before the fiftieth mission. But, on considering the matter, Jim concluded that the probability of a randomly selected pilot being able to fly 49 consecutive missions safely is $(49/50)^{49} = 0.3716017$. In other words, the odds are almost two to one against an ordinary pilot performing the feat that this pilot has already performed. Hence the pilot would seem to be more skillful than most and thus has a better chance of surviving the 50th mission. Who is right, Mia, Jim, or neither of them? Explain.

4. A fair die is rolled twice. Let A denote the event that the sum of the outcomes is odd, and B denote the event that it lands 2 on the first toss. Are A and B independent? Why or why not?

5. An urn has three red and five blue balls. Suppose that eight balls are selected at random and with replacement. What is the probability that the first three are red and the rest are blue balls?

6. Suppose that two points are selected at random and independently from the interval $(0, 1)$. What is the probability that the first one is less than 3/4, and the second one is greater than 1/4?

7. According to a recent mortality table, the probability that a 35-year-old U.S. citizen will live to age 65 is 0.725. (a) What is the probability that John and Jim, two 35-year-old Americans who are not relatives, both live to age 65? (b) What is the probability that neither John nor Jim lives to that age?

8. The Italian mathematician Giorlamo Cardano once wrote that if the odds in favor of an event are 3 to 1, then the odds in favor of the occurrence of that event in two consecutive independent experiments are 9 to 1. (He squared 3 and 1 to obtain 9 to 1.) Was Cardano correct?

9. Consider the four "unfolded" dice in Figure 3.9 designed by Stanford professor Bradley Effron. Clearly, none of these dice is an ordinary die with sides numbered 1 through 6. A game consists of two players each choosing one of these four dice and rolling it. The player rolling a larger number is the winner. Show that if all four dice are fair, it is twice as likely for die A to beat die B, twice as likely for die B to beat die C, twice as likely for die C to beat D, and *surprisingly*, it is twice as likely for die D to beat die A.

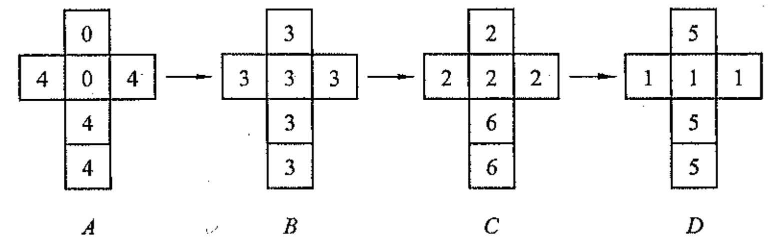

Figure 3.9 Effron dice.

10. **(Chevalier de Méré's Paradox†)** In the seventeenth century in France there were two popular games, one to obtain at least one 6 in four throws of a fair die and the other to bet on at least one double 6 in 24 throws of two fair dice. French nobleman and mathematician Chevalier de Méré argued that the probabilities of a 6 in one throw of a fair die and a double 6 in one throw of two fair dice are 1/6 and 1/36, respectively. Therefore, the probability of at least one 6 in four throws of a fair die and at least one double 6 in 24 throws of two fair dice are $4 \times 1/6 = 2/3$ and $24 \times 1/36 = 2/3$, respectively. However, even then experience had convinced gamblers that the probability of winning the first game was higher than winning the second game, a contradiction. Explain why de Méré's argument is false, and calculate the correct probabilities of winning in these two games.

11. In data communications, a message transmitted from one end is subject to various sources of distortion and may be received erroneously at the other end. Suppose that a message of 64 bits (a bit is the smallest unit of information and is either 1 or 0) is transmitted through a medium. If each bit is received incorrectly with probability 0.0001 independently of the other bits, what is the probability that the message received is free of error?

12. Find an example in which $P(AB) < P(A)P(B)$.

13. (a) Show that if $P(A) = 1$, then $P(AB) = P(B)$.
    (b) Prove that *any* event A with $P(A) = 0$ or $P(A) = 1$ is independent of *every* event B.

†Some scholars consider this problem to be the inception of modern probability theory.

14. Show that if an event A is independent of itself, then $P(A) = 0$ or 1.

15. Show that if A and B are independent and $A \subseteq B$, then either $P(A) = 0$ or $P(B) = 1$.

16. Suppose that 55% of the customers of a shoestore buy black shoes. Find the probability that at least one of the next six customers who purchase a pair of shoes from this store will buy black shoes. Assume that these customers decide independently.

17. Three missiles are fired at a target and hit it independently, with probabilities 0.7, 0.8, and 0.9, respectively. What is the probability that the target is hit?

18. In his book, *Probability* 1, published by *Harcourt Brace and Company*, 1998, Amir Aczel estimates that the probability of life for any one given star in the known universe is 0.00000000000005 independently of life for any other star. Assuming that there are 100 billion galaxies in the universe and each galaxy has 300 billion stars, what is the probability of life on at least one other star in the known universe? How does this probability change if there were only a billion galaxies, each having 10 billion stars?

19. In a tire factory, the quality control inspector examines a randomly chosen sample of 15 tires. When more than one defective tire is found, production is halted, the existing tires are recycled, and production is then resumed. The purpose of this process is to ensure that the defect rate is no higher than 6%. Occasionally, by a stroke of bad luck, even if the defect rate is no higher than 6%, more than one defective tire occurs among the 15 chosen. Determine the fraction of the time that this happens.

20. In a community of M men and W women, m men and w women smoke ($m \leq M, w \leq W$). If a person is selected at random and A and B are the events that the person is a man and smokes, respectively, under what conditions are A and B independent?

21. Prove that if A, B, and C are independent, then A and $B \cup C$ are independent. Also show that $A - B$ and C are independent.

22. A fair die is rolled six times. If on the ith roll, $1 \leq i \leq 6$, the outcome is i, we say that a *match* has occurred. What is the probability that at least one match occurs?

23. There are n cards in a box numbered 1 through n. We draw cards successively and at random with replacement. If the ith draw is the card numbered i, we say that a match has occurred. (a) What is the probability of at least one match in n trials? (b) What happens if n increases without bound?

24. In a certain county, 15% of patients suffering heart attacks are younger than 40, 20% are between 40 and 50, 30% are between 50 and 60, and 35% are above 60. On a certain day, 10 unrelated patients suffering heart attacks are transferred to a county hospital. If among them there is at least one patient younger than 40, what is the probability that there are two or more patients younger than 40?

25. If the events A and B are independent and the events B and C are independent, is it true that the events A and C are also independent? Why or why not?

26. From the set of all families with three children *a family* is selected at random. Let A be the event that "the family has children of both sexes" and B be the event that "there is at most one girl in the family." Are A and B independent? Answer the same question for families with two children and families with four children. Assume that for any family size all sex distributions have equal probabilities.

27. An event occurs at least once in four independent trials with probability 0.59. What is the probability of its occurrence in one trial?

28. Let $\{A_1, A_2, \ldots, A_n\}$ be an independent set of events and $P(A_i) = p_i$, $1 \leq i \leq n$.

(a) What is the probability that at least one of the events $A_1, A_2, \ldots, A_n$ occurs?

(b) What is the probability that none of the events $A_1, A_2, \ldots, A_n$ occurs?

29. Figure 3.10 shows an electric circuit in which each of the switches located at 1, 2, 3, 4, 5, and 6 is independently closed or open with probabilities p and $1 - p$, respectively. If a signal is fed to the input, what is the probability that it is transmitted to the output?

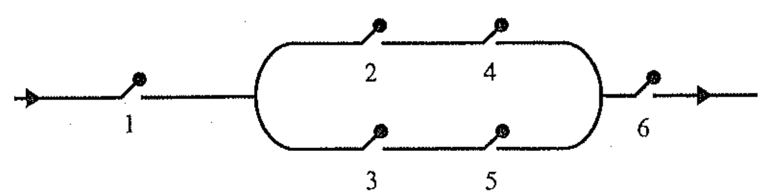

Figure 3.10 Electric circuit of Exercise 29.

B

30. An urn contains two red and four white balls. Balls are drawn from the urn successively, at random and with replacement. What is the probability that exactly three whites occur in the first five trials?

31. A fair coin is tossed n times. Show that the events "at least two heads" and "one or two tails" are independent if $n = 3$ but dependent if $n = 4$.

32. A fair coin is flipped indefinitely. What is the probability of (a) at least one head in the first n flips; (b) exactly k heads in the first n flips; (c) getting heads in all of the flips indefinitely?

33. If two fair dice are tossed six times, what is the probability that the sixth sum obtained is not a repetition?

34. Suppose that an airplane passenger whose itinerary requires a change of airplanes in Ankara, Turkey, has a 4% chance of losing each piece of his or her luggage independently. Suppose that the probability of losing each piece of luggage in this way is 5% at Da Vinci airport in Rome, 5% at Kennedy airport in New York, and 4% at O'Hare airport in Chicago. Dr. May travels from Bombay to San Francisco with one piece of luggage in the baggage compartment. He changes airplanes in Ankara, Rome, New York, and Chicago.

(a) What is the probability that his luggage does not reach his destination with him?

(b) If the luggage does not reach his destination with him, what is the probability that it was lost at Da Vinci airport in Rome?

35. An experiment consists of first tossing a fair coin and then drawing a card randomly from an ordinary deck of 52 cards with replacement. If we perform this experiment successively, what is the probability of obtaining heads on the coin before an ace from the cards?
Hint: See Example 3.31.

36. Let S be the sample space of a repeatable experiment. Let A and B be mutually exclusive events of S. Prove that, in independent trials of this experiment, the event A occurs before the event B with probability $P(A)/[P(A)+P(B)]$.
Hint: See Example 3.31; this exercise can be done the same way.

37. In the experiment of rolling two fair dice successively, what is the probability that a sum of 5 appears before a sum of 7?
Hint: See Exercise 36.

38. An urn contains nine red and one blue balls. A second urn contains one red and five blue balls. One ball is removed from each urn at random and without replacement, and all of the remaining balls are put into a third urn. If we draw two balls randomly from the third urn, what is the probability that one of them is red and the other one is blue?

39. Suppose that $n \geq 2$ missiles are fired at a target and hit it independently. If the probability that the ith missile hits it is p_i, $i = 1, 2, \ldots, n$, find the probability that at least two missiles will hit the target.

40. From a population of people with unrelated birthdays, 30 people are selected at random. What is the probability that exactly four people of this group have the same birthday and that all the others have different birthdays (exactly 27 birthdays altogether)? Assume that the birthrates are constant throughout the first 365 days of a year but that on the 366th day it is one-fourth that of the other days.

41. Figure 3.11 shows an electric circuit in which each of the switches located at 1, 2, 3, 4, and 5 is independently closed or open with probabilities p and $1-p$, respectively. If a signal is fed to the input, what is the probability that it is transmitted to the output?

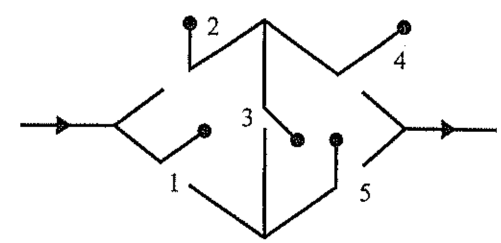

Figure 3.11 Electric circuit of Exercise 41.

42. In a contest, contestants A, B, and C are each asked, in turn, a general scientific question. If a contestant gives a wrong answer to a question, he drops out of the game. The remaining two will continue to compete until one of them drops out. The last person remaining is the winner. Suppose that a contestant knows the answer to a question independently of the other contestants, with probability p. Let CAB represent the event that C drops out first, A next, and B wins, with similar representations for other cases. Calculate and compare the probabilities of ABC, BCA, and CAB.

43. Hemophilia is a hereditary disease. If a mother has it, then with probability 1/2, any of her sons independently will inherit it. Otherwise, none of the sons becomes hemophilic. Julie is the mother of two sons, and from her family's medical history it is known that, with the probability 1/4, she is hemophilic. What is the probability that (a) her first son is hemophilic; (b) her second son is hemophilic; (c) none of her sons are hemophilic?
Hint: Let H, H_1, and H_2 denote the events that the mother, the first son, and the second son are hemophilic, respectively. Note that H_1 and H_2 are **conditionally independent** given H. That is, if we are given that the mother is hemophilic, knowledge about one son being hemophilic does not change the chance of the other son being hemophilic. However, H_1 and H_2 are not independent. This is because if we know that one son is hemophilic, the mother is hemophilic and therefore with probability 1/2 the other son is also hemophilic.