Final Study Guide 04/28/2011

A. Probability spaces, outcomes, events (discrete case)

1. What is the probability that a hand of 5 cards from a standard deck:
   a) contains exactly two aces?
   b) contains all cards of same suit?
   c) contains no pairs?
2. An experiment has 10 possible outcomes, labelled $0, 1, 2, \ldots, 9$, with probabilities $p(0), p(1), \ldots, p(9)$.
   a) What is the probability of the event that the outcome is greater than 5?
   b) If the outcomes are equally probable, what is $p(0)$?
   c) If the outcomes are equally probable and the experiment is performed three times, with each trial independent of the others, what is the probability that the three outcomes are all different?

B. Special Discrete Random Variables

1. Bernoulli
   a) A fair coin is flipped 10 times. Let X be the number of heads obtained. What is the probability that $2 \leq X \leq 8$?.
   b) A fair die is rolled 6 times. Let X be the number of 2s obtained. What is the probability that $2 \leq X \leq 4$?
   c) If an experiment is successful one in 10 times on average, what is the probability of exactly 10 successes in 100 trials? At least 10 successes in 100 trials?
2. Geometric
   a) If the probability of success on a single trial is p, what it the probability that the first success is on the n^{th} trial?
   b) What is the expected number of trials until the first success? I.e., what is $\mathrm{E}(X)$ if X is geometric with parameter p?
   c) What is the probability that the second success occurs on the 3^{rd} trial?
3. Hypergeometric
   a) There is a collection of N items, of which S are special. If n items are taken randomly without replacement. What it the probability that s of them are special?
   b) There is a collection of N fruits, of which a are apples and b are bananas. If n fruits are taken randomly without replacement and every fruit has an equal chance of being taken, then what it the probability that x apples and y bananas are taken? Let X is the number of apples taken and let Y be the number of bananas taken. Are X and Y independent? Why or why not?
4. Poisson. A discrete random variable X is Poisson with parameter λ, or Poisson(λ), if its pmf is

$$p(x|\lambda) = e^{-\lambda}\frac{\lambda^x}{x!}\,,\ x = 0, 1, 2 \ldots .$$

(Comment: Recall from calculus that $e^\lambda = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$, so the pmf of the Poisson random variable satisfies $\sum_{x=0}^{\infty} e^{-\lambda}\frac{\lambda^x}{x!} = 1$, as it must.) If an event occurs randomly in some continuous medium (e.g., time, space) with an average rate of λ occurrences per unit (time or space), and X is a random variable giving the number of occurrences in a unit, then X is Poisson(λ). (See page 206 of Ghahramani for a precise statement of what it means to for events to "occur randomly in time.")

a) Customers visit a web site randomly, regardless of time of day, at an average rate of 13 per hour. What is the probability that there are fewer than 300 visitors in a single 24-hour period. Express as a sum, but do not evaluate. (Hint: Change unit. What is the average number of customers per day?)

b) Compute $\mathrm{E}(X)$ if X is Poisson(λ) from the formula for the pdf.

c) Suppose X is Poisson λ and Y is Poisson μ. Assume X and Y are independent. Show that $X+Y$ is Poisson $\lambda+\mu$. Hint: $P(X+Y=n) = \sum_{i=0}^{n} p(i|\lambda)p(n-i|\lambda)$. Recall that $(a+b)^n = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} a^i b^{(n-i)}$.

C. Probability spaces, outcomes, events (continuous case)

1. Suppose X is uniform on $[a, b]$. What is the pdf of X? What is the cdf (cumulative distribution function) of X? Compute $\mathrm{E}(X)$. Compute $\mathrm{E}(X^2)$. Compute $\mathrm{Var}(X)$.
2. Suppose X has pdf cx^2 on $[a, b]$. What is c? What is the cdf of X? Compute $\mathrm{E}(X)$. Compute $\mathrm{E}(X^2)$. Compute $\mathrm{Var}(X)$.
3. Give an example of a continuous random variable X for which $\mathrm{E}(X)$ does not exist.
4. The normal random variable with mean μ and variance σ^2 has pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}.$$

The grades of 1000 students are (approximately) normally distributed with mean 82 and standard deviation 8. How many students (roughly) score in the 90s? In the 80s? In the 70s? In 60s?

5. The exponential random variable with parameter λ has pdf

$$f(x) = \lambda e^{-\lambda x}, \ x \in [0, \infty).$$

If events occur randomly in time (as on page 206 of Ghahramani), with λ events per unit time on average, then the time one waits (from a random starting time) until one event occurs is exponential with parameter λ.

a) Show (using the definition of expected value) that if X is exponential with parameter λ, then $\mathrm{E}(X) = \frac{1}{\lambda}$. Given the interpretation, why is this reasonable?

b) How long do you wait on average for the n^{th} event to occur?

c) If earthquakes occur at random times with an average rate of 3 per year, what is the probability that the next earthquake is at least 3 months away?

D. Change of variables

1. For univariate distributions

a) Suppose Z has pdf $f(z)$. Let $Y = aZ + b$. What is the pdf of Y?

b) Suppose Z has pdf $f(z)$. Let $X = Z^2$. What is the pdf of X?

c) What is the answer to these questions if

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2},$$

(i.e., Z is standard normal)?

d) Suppose X has pdf $f(x) = 2x$ on $(0, 1)$. Find the pdf of $1/X$.

2. For bivariate distributions

a) Suppose X and Y are independent exponential random variables with parameters λ and μ. Let $U = X + Y$ and $V = X/Y$. Find the joint pdf of U and V.

b) For the same X and Y, find the pdf of $W = XY$.

E. Pairs of random variables

1. Suppose X and Y are jointly distributed as follows:

		X		
		1	**2**	**3**
	1	.0	.1	.2
Y	**2**	.1	.2	.1
	3	.2	.1	.0

a) Find the marginal pmfs $f_X(x)$ and $f_Y(y)$. Find the pmf of $Z = X + Y$.

b) Compute the following: $\mathrm{E}(X)$, $\mathrm{E}(X^2)$, $\mathrm{E}(XY)$, $\mathrm{Var}(X)$, $\mathrm{Cov}(X, Y)$.

c) Are X and Y independent? Why or why not?

2. Suppose X is uniform on $[0, 2]$, Y is uniform on $[0, 3]$ and X and Y are independent.

a) What is the joint pdf $f_{X,Y}(x, y)$?

b) Compute $P(Y < X)$. Compute $P(\max(X, Y) < 1)$.

3. Suppose X and Y have joint pdf

$$f_{X,Y}(x, y) = \begin{cases} c\,(x^2 + y), & \text{if } 0 \le x \le 1 \text{ and } 0 \le y \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

a) What is c?

b) Find the marginal pdfs $f_X(x)$ and $f_Y(y)$.

c) Compute $\mathrm{E}(X)$, $\mathrm{E}(Y)$, $\mathrm{E}(X + Y)$, $\mathrm{E}(X^2)$, $\mathrm{Var}(X)$, $\mathrm{E}(Y^2)$, $\mathrm{E}(XY)$, $\mathrm{Var}(X + Y)$, $\mathrm{Cov}(X, Y)$.

d) Are X and Y independent? Why or why not?

e) Compute $P(Y < X)$.

4. Let b be a positive number. Suppose X and Y are jointly distributed uniformly on the triangle with vertices at $(0, 0)$, $(1, 0)$ and $(0, b)$ (i.e., the joint density $f(x, y)$ is constant

on this triangle and 0 elsewhere else). Find the following (showing work): $f_X(x)$, $\mathrm{E}(X)$, $\mathrm{E}(XY)$, $\mathrm{Cov}(X,Y)$, $f_{X|Y}(x|y)$, $\mathrm{E}(X^2|Y=y)$. Are X and Y independent? Why or why not?

F. Generating functions and moment generating functions.

1. Show that for *any* independent random variables, $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$ (provided the functions are defined on some open interval containing 0).
2. Let X be a Poisson random variable with parameter λ. Let Y be a Poisson random variable (independent of X) with parameter μ.
   a) Find the moment generating functions $M_X(t)$ and $M_Y(t)$.
   b) Find the first and second derivatives of $M_X(t)$ with respect to t, evaluated at 0 (i.e., find $M_X'(0)$ and $M_X''(0)$).
   c) Show that the product $M_X(t) \cdot M_Y(t)$ is the moment generating function of a Poisson random variable with parameter $\lambda + \mu$.
   d) Compare with B.4.c.
3. Find the moment generating function of the exponential random variable. Determine the pdf of $X + Y$ if X is exponential λ and Y is exponential μ.