Content Summary
Ratio, Proportion, Linear Functions and Functions in General

James J. Madden
LSU Department of Mathematics

0. Introduction

Ratio and proportion is an ancient topic that is treated in great detail in Euclid's *Elements* (300 B.C.). It has been a part of the school curriculum since the earliest "modern" textbooks. For example, Dilworth's *Schoolmaster's Assistant* (1796) includes a description of the so-called "Rule of Three", which directs one to find the fourth term in a direct proportion by muliplying the second and third terms and dividing by the first—a skill that merchants would use frequently. (If 12 loaves cost \$15, how much will 20 cost?) Today, proportional reasoning studied for reasons that go well beyond its practical usefulness. It is a transitional topic, coming between arithmetic and algebra.

Underlying the progression from proportional reasoning through algebra and beyond is the idea of a function. One may view proportional reasoning as a point of entry to the function concept. In its barest abstract mathematical definition, a function is a rule that associates with each element of one set—called the domain—a unique element of another set, which, in recent technical writing, has come to be called the "codomain". Modern mathematics shows that the concept of a function is a remarkably versatile, fundamental notion. In school mathematics, the concept of a function is both an abstract mathematical idea and also a schema for organizing and structuring instruction. A collection of domain-codomain pairs determined by a function is a kind of data set. It can be displayed as a table or plotted as a graph. Because a function follows a rule, patterns determined by that rule are almost always visible in the table or graph. The patterns can be described in words or by equations, and they can be associated with concrete situations. The representations that occur over and over in the mathematics curriculum—patterns, verbal descriptions, equations, graphs and tables—all hang together about the function concept.

1. Ratio, Rate and Proportion

It is useful to begin with some history. Euclid defined "ratio" as follows:

> *A ratio is a sort of relation in respect of size between two magnitudes of the same kind.* (*Elements*, Book V, Definition 3.)

For Euclid, a ratio was a direct comparison between magnitudes (*e.g.*, lengths, areas, volumes, *etc.*) No measuring was involved. This is why he required the magnitudes compared to be "of the same kind". (How would you compare a length with an area, for example, without measuring?)

In modern times, we typically measure the magnitudes we encounter. (Measurement itself is a deep idea—far deeper than most people assume. Some additional discussion of measurement appears in Appendix C.) By means of measurement, we attach numbers to all of them. Today, we conceive of a ratio as a comparison between two numbers obtained by measuring two magnitudes of the same kind using the same unit. Often, a ratio is expressed as a single number, namely the one obtained by dividing the measure of the first magnitude by the measure of the second. This number is independent of the unit used.

By means of measurement, we can go beyond Euclid. We are able to compare *any* two magnitudes without concern for their kind by comparing the numbers obtained by measuring each with some selected unit. A comparison like this is called a "rate". When describing rates, we must communicate the kind of measure and the units used, of course.

Comparing by counting. As mentioned in the Introduction, one pedagogical strategy of the Louisiana Comprehensive Curriculum is to introduce abstract ideas by examining them in concrete contexts. Perhaps the most concrete class of instances of the ratio concept occurs in the comparison of numbered collections of objects such as beans, pennies, blocks or tiles. To compare the sizes of two piles, count the number of objects in each pile. For example, there might be 4 pebbles in the first pile and 7 in the second. Then we say that the ratio of the sizes of the piles, the first to the second, is $4 : 7$.

Other pile-pairs with the same ratio can be obtained by combining copies of the original pair. To make this idea clear, you might make and display a model with some specific counts, *e.g.*, 4 and 7. Then build numerous pairs of piles, each identical with the model pair. If you combine pairs by first lumping together some or all the piles with 4 objects and then lumping together the corresponding 7-objects piles, you get a pair of large piles. The relative sizes of the two large piles is the same as the relative sizes of the components. You can divide each of the large piles in the same way—in half, or in thirds or quarters or take equal portions of each. The relative sizes of corresponding partial piles are the same. Pairs created this way are said to be *proportional.*

Ratios and rates in general. As we suggested above, ratios and rates are numbers that are used to describe the relationship between measured magnitudes. Thus, the basic operation involved in finding ratios and rates is to measure and divide. Suppose we have two magnitudes A and B. If we measure A and B with appropriate units, we get two numbers. They represent properties of A and B. (The numbers and their meanings depend, of course, on the units used. Are they miles? Hours? Minutes?) If we divide the number that goes with A by the number that goes with B, we get a new number that shows something about the relationship of A to B. Like the measurements, this number and its meaning also depend, in general, on the units used.

Ratio is a special case. If A and B are of the same kind and the same unit is used for measuring both is the same, then the choice of unit does not matter. The number we get upon division is the same, regardless of the unit chosen, provided the same unit of measure is used for both A and B. We call this number the ratio of A to B (with respect to the magnitude of interest). *Example.* Suppose A is a trip from Baton Rouge to El Paso, Texas and B is a trip from El Paso to Las Cruces, New Mexico. The first trip measures 1019 *miles* and the second measures 47 *miles.* If we divide the 1019 *miles in trip A* by the 47 *miles in trip B*, we get 1019/47, or about 21.68. This number tells us the number of miles in trip A for each mile in trip B. It is *the ratio of the length of trip A to the length of trip B*, or the *ratio of A to B with respect to length.* We could have measured both distances in kilometers or feet or inches. The ratio of the length of trip A to the length of trip B would be the same. Note that we could have considered other features of the trips: the time each took, the amount of gas or the cost. We could have found the ratio of the trips with respect to cost, the ratio of the amount of gas used in each, *etc.* These might all be different.

Rates are more versatile. We can divide one measurement by another even when the kinds of magnitude involved and the units used to make the measurement are different. We reserve the term "ratio" for a comparison of like magnitudes and we use the term "rate" for a comparison of unlike quantities (*e.g.* miles per hour). In this case, it is essential to communicate the units that were used for the measurement. Phrases such as "miles per hour", "cost per pound", "mass per cubic centimeter" are commonly used. *Example.* Suppose A is the distance we drove one afternoon and B is the time it took. Let's say our trip measured 72 *miles* and the time was 54 *minutes*. If we divide the 72 *miles* by the 54 *minutes* we get 72/54 *miles per minute* or more simply 4/3 *miles per minute.*

In both cases—ratios and rates—we occasionally forgo the division and record the numbers and their meanings themselves. One might say, for example that *the ratio of the lengths of the two trips A and B was* $1017 : 47$. In the same vein, the meaning of saying *my speed was 72 miles per 54 minutes* is perfectly clear, albeit unusual.

We would like to suggest some rules for clear understanding and clear communication. When using measurements, we always label our numbers (as pounds or inches or dollars or whatever is appropriate) in order to show what they mean. Without labels, the numbers mean nothing. The same is true when talking about rates. The usual labels employ the word "per". When speaking of ratios, in contrast, it is not important to quote a unit, but it is essential to say whether the ratio is with respect to length or height or weight, *etc.*. It would be nonsense to say that the ratio of John to Tom was 1.2. Is this a ratio of weights? Heights? Ages? Income? Or what?

Similar triangles. Suppose we have two triangles: $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$. Each side of the first triangle has a ratio (with respect to length) with the corresponding side of the second: $A_1B_1 : A_2B_2$, $B_1C_1 : B_2C_2$, $C_1A_1 : C_2A_2$. In each pair, we have a side from the first triangle and the corresponding side form the second. If all three of these ratios are the same, we say the triangles are *similar.* In this case, the common value is a *scaling* *factor*.

Functions associated with ratios and rates. In many situations we are concerned not with just one pair of measurements and the associated ratio (or rate), but with *many* pairs that come in a class. Thinking about ratios and rates in terms of such classes links ratios and rates to functions.

Imagine gathering data on the height of various objects and the lengths of their shadows. For each pair, you can form the ratio, "length of shadow divided by height". These ratios will vary, depending on the time of day that the measurements were made. For pairs collected at noon the ratio is small since the shadow is short. For those pairs gathered at dusk, the ratio is large, since the height is then much less than the length of the shadow. Thus, in general, there will be no way to predict the length of the shadow from a knowledge of the height of the object only. However, if you measure numerous heights and shadows on flat land at the same time of day, then the ratios will all be the same. In this case, if you know the ratio of one pair you know the ratio of all the other pairs. Knowing the common ratio enables you to determine the shadow length from the height. It forms a rule relating the two measurements. Shadow length, under these circumstances, is a function of height.

For another example—this time pertaining to rates—suppose you examine the receipts obtained when people buy gas for their cars. Each slip shows a number of gallons purchased and a total paid for gas (excluding tax). For each slip you may compute a rate, the price per gallon. If you look at numerous receipts acquired by different drivers at the same gas pump on the same day, you will see the same rate. In this case, knowing this one rate enables you to calculate the total paid from the number of gallons purchased. The total paid is a function of the number of gallons purchased. On a different day, or at a different pump, the function that relates number of gallons purchased to total paid may be different. This, of course, is just a complex way of saying that the relationship of what you purchase to what you pay depends upon the price per gallon, which may vary from day to day or dealer to dealer. (This paragraph is intended to illustrate the use of the mathematical language in a situation that you are familiar with. The language seems clumsy and long-winded in familiar situations. But in new or unfamiliar situations, it provides clarity. Of course, to believe this, you probably need to experience it.)

Further examples abound. When you change the size of a recipe, the amount of each ingredient changes by the same factor. That is, if there are 3 times as many eggs in the new recipe, then there is also 3 times as much flour. The ratio of the amount of each ingredient in the new recipe to the amount in the original is the same. For each ingredient, there is an "old amount" and a "new amount", and the new amount is obtained from the old by the same rule. Or consider the pairs of measurements that arise when you enlarge a photograph. The pairs involved here include a length measurement of some item in the original photo and a length measurement of the same item in the new photo. All the linear dimensions change by the same factor. That factor–often called the *scaling factor*—is the ratio of each new length to the corresponding length in the old photo. For an example with rates, if you move at a constant speed, then the rate of distance traveled to time elapsed is the same for each portion of your trip. If you purchase different quantities of some item at the same price per unit, then the amount purchased divided by the cost remains the same. In each case, there is a function.

Important examples arise with classes of similar triangles. Suppose we have numerous triangles $\triangle A_1B_1C_1,\ \triangle A_2B_2C_2,\ \triangle A_3B_3C_3, \ldots$ that are *all* similar to one another. This means, in particular, that for any i,

$$\frac{B_iC_i}{B_1C_1} = \frac{A_iB_i}{A_1B_1}.$$

Now multiply both sides of this equation by B_1C_1. We get:

$$B_iC_i = \frac{B_1C_1}{A_1B_1} A_iB_i.$$

This shows that in each triangle, the length B_iC_i can be calculated from the length A_iB_i. You multiply the latter by the ratio $\frac{B_1C_1}{A_1B_1}$. Again, we have a function. If we divide both sides of the last equation by A_iB_i, we get:

$$\frac{B_iC_i}{A_iB_i} = \frac{B_1C_1}{A_1B_1}.$$

This common ratio is not a scaling factor, but a property of the triangle.

Yet more examples come in situations such as the following:

- Shopping: unit costs. (At one store, a dozen eggs cost \$1.08. At another store, 6 eggs cost \$.53.)
- Travel: speed, gas mileage, cost of vehicle. (You travel 310 miles after filling up. To refill the your tank, it takes 14.3 gallons.)
- Finance: interest rates on savings accounts, mortgages, and credit cards.
- Conversions: inches, feet, meters, etc.
- Recipes and mixtures.
- Geometry (both applied and theoretical). Stairs, shadows, similar triangles.

Proportional reasoning. Proportional reasoning, as a mental activity, involves recognizing, creating, examining and representing one or more families of pairs of numbers that have similar interpretations and exhibit the same ratio or rate. Interesting and sophisticated activities often involve making multiple representations, transforming one representation into another or choosing an appropriate representation for a given purpose. Many textbooks devote substantial amount of time to other problems of a more routine procedural nature, such as determining a ratio (or rate) from a given pair of measurements or determining what one of the numbers in a pair must be when the other is known and the pair is known to belong to a family with a given ratio (or rate).

The most important representations are:

a) *Verbal descriptions.* We have already given a few examples.

b) *Pictures, diagrams or models.* There is a lot of variety here; see the activities in the TEXTEAMS Proportional Reasoning Workshop.

c) *Tables.* Tables effectively convey several ideas that are important in proportional reasoning. It is common to write the pairs as rows. The columns correspond to the roles occupied by the members of each pair. A table may have more than just two columns, of course. One might want a column in which to write the ratio of each pair (particularly if pairs in different ratios appear in the table). A table displaying numerous pairs *in the same ratio (or rate)* exhibits many striking patterns. In such a table, for example, if you make a new pair by adding or subtracting rows, it will have the same ratio (or rate) as the other pairs in the table.

d) *Graphs.* By standard conventions, any pair of numbers corresponds to a point in the coordinate plane, with the first number giving the x-coordinate and the second giving the y-coordinate. Thus, a family number pairs corresponds to a collecting of points. If the number pairs represent numerous instances of the same ratio (or rate), they lie on a line that passes through the origin. The slope of the line is the ratio

e) *Equations.* As we have seen, in proportional reasoning we are always dealing with equations of the form $y = ax$. Here, a represents the common ratio that occurs in a class of proportional pairs.

In the Louisiana Comprehensive Curriculum, there are several important pedagogical expectations related to functions:

A) The curriculum is divided into units. Each unit has a small number of explicit dominant themes an a number of incidental themes. Specific classes of functions are dominant themes in several units, and functions are incidental themes in many others.

B) The curriculum takes students into situations that are close to common experiences in which these functions are present; students learn to talk about functions in everyday language first and gradually progress to specialized mathematical terminology.

C) Students learn multiple representations for functional relationships—*e.g.*, verbal descriptions, models, tables, graphs, equations—and they gain significant experience translating from one representation to another and describing the systematic relationships and analogies between different representations.

D) Students are expected to learn certain specialized language and procedures concerning the functions of each type.

As in other curricula, there tends to be a progression. Proportional functions are introduced and studied earlier than linear functions. Quadratic functions follow linear functions and other increasingly complex functions come after that. The language and techniques become increasingly modern as the functions become more complex. Much of the language concerning proportion can be traced clearly back to Euclid; linear and quadratic functions were not described by equations and graphed in the manner we presently do until the 17^{th} century, and much of the work that students do with these functions is intended to prepare them for calculus.

Appendix A: Functions.

A. Ways of representing functions
   - By a **table** of data consisting of a family of input-output pairs. Generally, there are infinitely many possible input-output pairs. A table can never show more than a sample.
   - By an **equation**, involving an independent variable (usually x) that stands for input, and a dependent variable (usually y) that stands for output
   - By a **graph**. In the coordinate plane, each number pair corresponds to a position. You make the graph of a function f by darkening or coloring all the points corresponding to the input-output pairs of f.
   - By a **colloquial description** using words and/or diagrams.[1]

B. Kinds of functions. (In these illustrations, a, b, c, and d are constants; x and y are the independent and dependent variables, respectively.)
   - Constant: $y = a$.
   - Linear: $y = ax + b$. (Proportional functions are linear functions of the form $y = ax$.)
   - Quadratic: $y = ax^2 + bx + c$.
   - Other. (For example, cubic functions are functions of the form $y = ax^3 + bx^2 + cx + d$. Exponential functions are functions of the form $y = b^x$, where b is a positive constant.)

The linear and quadratic functions figure most prominently in the $6^{th} - 10^{th}$-grade curriculum.

C. Pedagogy.
   - Before algebra, functions tend to be implicit in the curriculum. Proportionality is an important topic in middle school. In algebra, students begin using terminology and notation relating to functions. They see numerous examples of linear, quadratic and exponential functions and study them in detail using the various representations.

[1] This is often the way a function appears in a problem. *E.g.*, if you make cardboard boxes 2 feet high, 3 feet from front to back and of variable width, and if they have double thickness on the top and the bottom and single thickness on the sides, then the amount of cardboard used is a function of the width. What is the largest possible width if no more than 44 square feet of cardboard may be used?

Appendix B: Proportional Relationships.

- A proportional relationship is a special kind of linear function. However, proportional reasoning is used in a wide variety of settings in which functional nature of the concept is not a major focus of attention. For example, in calculating the tip on a restaurant bill, you may view the cost of food and beverage as input (independent variable) and the tip as a dependent variable. Possibly the best setting for introducing proportional relationships to students is by displaying and discussing **proportional families**. A proportional family is a collection of pairs of magnitudes that all have the same ratio[2]; for example, $\{ (3,7), (6,14), (9,21), \ldots \}$. Proportional families arise in numerous contexts in mathematics and in practical life.
- Proportional relationships can be described colloquially and by tables, equations and graphs.
- In geometry, proportionality is seen in similar figures. In this context, two different kinds of proportional families arise:
    - If two figures are similar, then the pairs consisting of (1) the measure of any segment in the first figure followed by (2) the measure of the corresponding segment in the second figure gives a proportional family. The entries in each pair have the same ratio. From a functional point of view, you may regard the linear measures in the first figure as inputs and the corresponding measures in the second as outputs.
    - Given a whole family of similar figures, then the pairs consisting of (1) the measure of a particular segment (say, the shortest side) in *any* figure followed by (2) the measure of another particular segment in the same figure (say, the longest side) gives a proportional family. Once again, entries in each pair have the same ratio. The difference is that in the first instead, only two figures are considered, but all parts are measured and compared. In the second, many figures are considered, but in each only two parts are measured and compared. In the first instance, the comparisons are between figures. In the second they are within figures.

[2] We agreed to use Euclid's definition of ratio: *a relationship between like magnitudes with respect to size.* Unlike Euclid, we routinely associate numbers with ratios. Fractions and percents are ways of expressing numbers.

Appendix C: Measurement.

What does measurement mean? When we measure a magnitude A, we do two things:

1) we compare A to a standard U of the same kind—if A is a length, then U is also a length; A might be your height and U a meter stick;
2) we somehow find a number that bears the same ratio to the number 1 as A bears to U; this is called the measure of A in units U.

Note that in explaining what measurement means, we must think of ratios in the same way that Euclid did—namely as direct comparisons—for the ratio of A to U cannot be treated as a number until *after* we measure. Therefore, in our explanation of what measurement means, we need to understand how two ratios, viewed not as numbers but as direct comparisons, can be the same. Euclid supplied a definition that solves this problem. Definition 5 of Book V may be paraphrased as follows: *the ratio of A to B is the same as the ratio of C to D if, for all positive integers k and ℓ, $kA \leq \ell B \iff kC \leq \ell D$.* (Here kA means the magnitude we get by putting together k copies of A.)

This explains the principle, but not the practical method. Let us examine the process of measurement in detail. Suppose A and U are magnitudes. If we want to measure A with U, then we might start out by asking how many times we can fit U inside of A without overlaps. The idea of "overlaps" being somewhat vague, it would be more precise to ask how many times we can *subtract* U from A until we are either left with nothing or with something smaller than U. If U can be subtracted exactly 6 times (say) with nothing left over, then we say the measure of A in units U is 6. If there is a remainder, then we still have an estimate with error bounds. If U can be subtracted 6 times (say) with some amount R that is smaller that U left over, then we say the measure of A in units U is more than 6 but less than 7.

To get any additional information about the measure of A, we need to measure R. We can learn nothing new about R by attempting to measure it with U. We already know that R is smaller than U. We must compare R with a magnitude that is smaller than U. To do this, we divide U into a number of equal magnitudes, and measure R with one of these. When we are measuring in inches (*i.e.*, when U is the standard inch), for example, is usual to use $1/4^{th}$s or $1/16^{th}$s or $1/32^{nd}$s of U, depending on the amount of accuracy needed. In the metric system, on the other hand, it is customary to divide the unit into 10 equal parts.

Suppose we are using meters, and suppose we have found the length of A to be between 6 and 7 meters in the first step. We create a new unit U_1 equal to exactly one tenth of U and we measure the remainder with that. We might find that R is exactly 4 U_1s. Or we might find, perhaps that the remainder is longer that 4 U_1s but not yet 5 U_1s. Then we conclude that the length of A is between 6.4 and 6.5 meters. Having subtracted 6.4 meters from A, we are left with a second remainder, R_2. We subdivide U_1 into ten equal parts, each a hundredth of a meter long. We select one of these and call it U_2. We use U_2 to measure the second remainder R_2. If there is a *still* a remainder, we measure that with thousandths. And so we continue with ten-thousandths, hundred-thousandths, *etc.*