**Complex Numbers: Some Help and Hints**

Student: *I think I've got it pretty much down up until 5.2. I understand (from the work we did on Monday) that if $T_bR_m(z) = mz + b$, then $b$ makes it translate (i.e., if $b = 0$, then it would just rotate) and if $m = x + yi$, then $x = 0$ for it to not dilate. But, I am at a loss as to how to show it. I can state it by citing the figure that we drew and then drew its image, but I would appreciate any helpful advice.*

JJM: Oh, no! It's not the condition $x = 0$ that enables you to say that $T_m$ does not dilate. Rather, it is the condition $|m| = 1$. If $|m| = 1$, then $T_m$ will produce no dilation—only rotation. Why is that? If $m = x + iy$, and $|m| = 1$, then $x^2 + y^2 = 1$, so $x = \cos(\theta)$ and $y = \sin(\theta)$ for some $\theta$. This $\theta$ is the measure of the rotation. You can convince yourself of this by reasoning as follows. First, $T_m$ is an isometry (because $|m| = 1$). Second,

$$\begin{aligned} T_m(0) &= 0, \\ T_m(1) &= \cos(\theta) + i\sin(\theta), \\ T_m(i) &= -\sin(\theta) + i\cos(\theta). \end{aligned}$$

Finally, an isometry is completely determined by where is sends three non-colinear points. Put these three things together, and you see that $T_m$ is a rotation by $\theta$ radians about 0.

Student: *5.3 is easy with the proofs of 5.1 and 5.2.*

JJM: Yes, I think so.

Student: *6.1 is similarly easy because it is basically 5.1 again.*

JJM: Well, it's a little more. You also need 5.3, I think. Suppose $S(z) = mz+b$ and $T(z) = kz+a$ for some complex numbers $m$, $k$, $b$ and $a$. (We know that the transformations can be represented in this way from 5.3.) Then $ST(z) = S(T(Z)) = mT(z)+b = m(kz+a)+b = mkz + (ma + b)$. For this to be a translation, we must have $mk = 1$.

Student: *Which leaves 6.2 as another headscratcher. Once again, earlier I had just stated 5.2, I did not use the $\theta$ in the question at all. Should I just see that $i$ equals $\pi/2$ radians (I think) and see how each $m$ of $S$ and $T$ relates to the total rotation?*

JJM: Well, first of all, $i$ does not equal $\pi/2$ radians; $i$ is a number; it is not a measure of anything. Maybe you are thinking of the fact that

$$i = \cos(\pi/2) + i\sin(\pi/2).$$

(This is because $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$.) Your idea to "see how each $m$ of $S$ and $T$ relates to the total rotation" is right on track, however. You need to translate this idea into mathematical language. I did the critical piece of translating above. By 5.3, you may assume $S(z) = mz + b$ and $T(z) = kz + a$, for some complex numbers $m$, $k$, $b$ and $a$, with $|m| = 1$ and $|k| = 1$.

Student: *Similarly, for 6.3, I did not use $a + bi$, so I don't know how to determine the center.*

JJM: Refer back to what I said about 6.1.

Student: *I hope this doesn't make you think of me as a lost cause, but any hints that you could give would be greatly appreciated.*

JJM: You are certainly not a lost cause. I hope this helps.

Student: *Have a wonderful day and an early start on Spring Break.*

JJM: You, too—as soon as you finish your homework!