The Complex Numbers and Geometric Transformations

Adding, subtracting multiplying and dividing

Our first goal will be to introduce the complex number in an intuitive way, to enable you to perform the basic operations (add, subtract, multiply and divide) with them.[1]

The complex number system is built from three **assumptions**. We will treat them as axioms. (In a course in abstract algebra—like LSU M4200—these assumptions would be derived from yet more basic assumptions about algebraic systems.)

i) There a number whose square is -1. We call this number i.

ii) We can compute with i just as we would compute with a real number. In other words, we can add any real number to i and multiply i by any real number. Moreover, we can add and multiply any numbers we thus obtain or that we ever get by starting with $\mathbb{R}$ and i and adding and multiplying over and over.

iii) Addition and multiplication, even when i is involved, obey the familiar rules:
- a) $(x + y) + z = x + (y + z)$,
- b) $x + y = y + x$,
- c) $x + 0 = x$,
- d) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
- e) $x \cdot y = y \cdot x$,
- f) $1 \cdot x = x$,
- g) $x \cdot (y + z) = x \cdot y + x \cdot z$

Definition. The collection of all numbers we get by starting with $\mathbb{R}$ and i and using addition and multiplication over and over in any way is called *the field of complex numbers* and it is denoted $\mathbb{C}$.

Problem. Let $S = \{\, a + bi \mid a, b \in \mathbb{R} \,\}$. In other words, S is the set of all numbers that can be written as a real number plus a real multiple of i.

1.1. Prove that $\mathbb{R} \subseteq S$.

1.2. Prove that $i \in S$.

1.3. Prove that $S \subseteq \mathbb{C}$.

1.4. Prove that if you add together any two elements of S, the result is in S. (Hint: suppose $a + bi$ and $c + di$ are in S. Show that the sum of these two numbers can be written as a real number plus a real multiple of i.)

1.5. Prove that if you multiply together any two elements of S, the result is in S. (Hint: this is similar to the previous part, but the algebra is a little more involved.)

1.6. Conclude from 1.4, 1.5 and Assumption *ii*) above that $S = \mathbb{C}$.

Definition. Suppose that $z = u + vi$ is a complex number. The *complex conjugate of* z, denoted $\overline{z}$, is $\overline{z} := u - vi$.

Problem. Can you divide by a complex number? The following exercises show how:

[1] Note that the Louisiana GLEs include the following at the 11^{th}–12^{th}-grade level: *Read, write, and perform basic operations on complex numbers.*

2.1. Prove that $z\overline{z}$ is a real number. Since $z\overline{z}$ is real, $\frac{1}{z\overline{z}}$ has an obvious meaning. Show that $\frac{1}{z\overline{z}}\,\overline{z}$ is a complex number.
2.2. We say that w is the multiplicative inverse of z if $wz = 1$. If $z \neq 0$, show that the multiplicative inverse of z is $\frac{1}{z\overline{z}}\,\overline{z}$.
2.3. Write $\frac{3+4i}{5+7i}$ in the form $a + bi$, with a and b real.
2.4. Show how to write $\frac{a+bi}{c+di}$ in the form $u + vi$ with u and v real.

Definition. Suppose that z is a complex number. The *modulus of* z, denoted $|z|$, is $\sqrt{z\,\overline{z}}$.

Problem. We investigate the properties of the modulus.
3.1. Show that every complex number has a modulus and that the modulus is a positive real number.
3.2. Show that z has modulus 1 if and only if $z = \cos\theta + i\sin\theta$ for some angle θ.
3.3. Show that every nonzero complex number can be written as a positive real number times a complex number of modulus 1.

Geometry of Complex Numbers

Let $\mathbf{P}$ be a plane with an orthogonal coordinate system whose coordiante functions are x, y. We can make a bijection between the points in $\mathbf{P}$ and the complex numbers by associating to the point $P(x, y)$ the complex number $x + iy$. (We will switch to writing complex numbers with i first in the second term.)

Suppose we have done this. From now on, we will use $x + iy$ to refer either to a complex number or to a point in $\mathbf{P}$, without making an explicit distinction. We will call $\mathbf{P}$ with this labelling *the complex plane*.

Problem. We will show how to use modulus to talk about distance.
4.1. Show that the distance between points z and w is $|z - w|$.
4.2. Show that $|zw| = |z||w|$.
4.3. For a fixed complex number w, let $R_w : \mathbf{P} \to \mathbf{P}$ be the transformation that sends point z to point wz. Show that if $|w| = 1$, then R_w is an isometry.

Problem. We will show how to use complex numbers to talk about transformations.
5.1. For a fixed complex number w, let $T_w : \mathbf{P} \to \mathbf{P}$ be the transformation that sends point z to point $z + w$. Show that T_w is an isometry.
5.2. Let R be a rotation by an angle of θ radians about the point $a+ib$. Find complex numbers b and m so that $R(z) = T_b R_m(z)$. (Note that $T_b R_m(z) = mz + b$.
5.3. Show that every orientation-preserving isometry of $\mathbf{P}$ can be represented by a complex function of the form $f(z) = mz+b$, where m and b are complex numbers and $|m| = 1$.

Problem. We know that the composite ST of two orientation-preserving isometries T and S is an orientation-preserving isometry. Use the complex representation of isometries to answer the following questions:
6.1. Under what conditions on T and S is ST a translation?
6.2. If ST is a rotation, then what is its angle (in terms of the angles of S and T)?
6.3. If ST is a rotation, then what is its center (in terms of data about S and T)?