# Coordinates, Lines and Linear Equations

In this activity, you will develop an understanding of how to answer the following questions:

1. Given a linear equation (in two variables), why is its graph a line?
2. Given a line, why it there a linear equation of which it is the graph?

# Review of Coordinates

### Coordinates on a line

Given a line, a *one-dimensional coordinate system* on that line is made as follows:

1. First, make the following choices:
   - choose a point on the line and call it $O$;
   - choose a second point $P$ on the line and call $OP$ the "unit interval".
2. Using the unit interval as a unit of measure, assign to each point on the line the real number measure of its "oriented distance" from $O$. (The oriented distance between $O$ and a point $X$ different from $O$ is either positive or negative; it's positive if $P$ and X are on the same side of $O$ and negative if $P$ and $X$ are on different sides.)

**Fact 1.** A one-dimensional coordinate system constructed as above on a given line creates a one-to-one correspondence between the points on that line and the real numbers. The point $O$ corresponds to the number 0. If points $A$ and $B$ correspond to numbers $a$ and $b$, then the distance between $A$ and $B$, as measured by the unit interval $OP$ of the coordinate system, is $|b-a|$. (We will not try to prove this fact here. You may treat it as an axiom.)

### Coordinates on a plane

Given a plane, *a standard coordinate system* on it is made as follows:

1. First, make the following choices:
   - choose two <u>perpendicular</u> lines; call the point of intersection $O$; name one of the lines the "$x$-axis" and name the other the "$y$-axis";
   - choose points $P$ and $Q$ on the $x$- and $y$-axes, respectively, <u>with *OP* congruent to *OQ*</u>.
2. Make a one-dimensional coordinate system on each of the lines, using $OP$ as the unit interval on the $x$-axis and $OQ$ as the unit interval on the $y$-axis.
3. Given a point $A$ in the plane, assign coordinates to it as follows: Draw the line through $A$ parallel to the $y$-axis. This line crosses the $x$-axis. Determine the coordinate of this point in the one-dimensional system on the $x$-axis. This is called the $x$-coordinate of $A$ and is here denoted $x(A)$. The $y$-coordinate of $A$ is determined in the same way, except interchanging the roles of $x$ and $y$.

**Fact 2.** A standard coordinate system constructed as above on a given plane creates a one-to-one correspondence between the points in the plane and the set of ordered pairs of real numbers. (We will not try to prove this fact here. You may treat it as an axiom.)

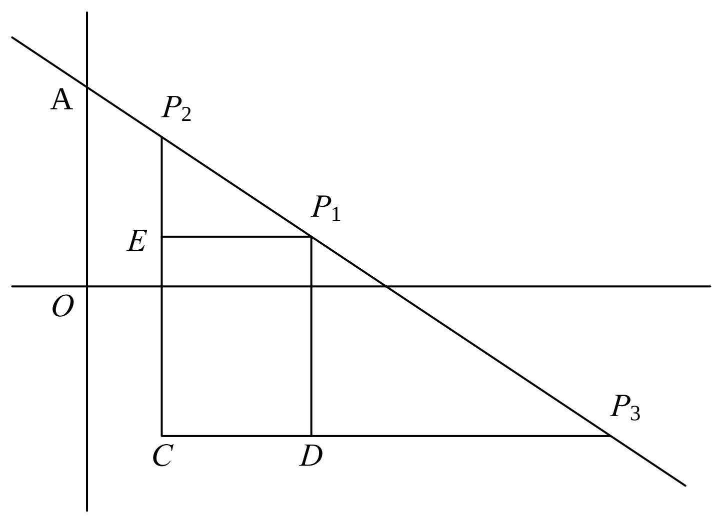
**Addendum.** The rule for passing *from a point to an ordered pair* of real numbers was described in B.3., above. The rule for going *from an ordered pair* of real numbers $(a, b)$ *to a point* is as follows: On the $x$-axis, locate the point with label $a$ and through it draw the line parallel to the $y$-axis; on the $y$-axis, locate the point with label $b$ and through it draw the line parallel to the $x$-axis. The point where these lines meet is desired point. It is denoted $P(a, b)$.

**Fact 3.** Suppose a coordinate system on a plane has been made. Suppose a line $l$, not parallel to the $y$-axis has been chosen. Suppose $P_1 = P(x_1, y_1)$, $P_2 = P(x_2, y_2)$ and $P_3 = P(x_3, y_3)$ are three different points on $l$. Then:

$$(y_2-y_1) / (x_2-x_1) = (y_3-y_1) / (x_3-x_1). \qquad (1)$$

*Proof.* Draw lines, as shown in the diagram, parallel to the axes. The argument on pages 48-49 shows that

$$| P_2E | / | EP_1 | = | DP_1 | / | P_3D | .$$




In terms of coordinates, this can be rewritten:

$$|y_2-y_1| / |x_2-x_1| = |y_3-y_1| / |x_3-x_1| .$$

This shows that the two sides of (1) are equal in absolute value. *It remains to show that the two sides of (1) have the same sign.* We consider cases. First, if $l$ is parallel to the $x$-axis, then both sides of (1) vanish. Otherwise, let $A$ be the point where $l$ crosses the y-axis, and choose a point $R$ on the $y$-axis with bigger $y$-coordinate than $A$ and a point $S$ on $l$ with positive $x$-coordinate. Then, if angle $RAS$ is less than a right angle, points on $l$ with larger $x$-coordinate also have larger $y$-coordinate. This means that both sides of (1) will be positive, since in each side the numerator and denominator will have the same sign. On the other hand, if angle $RAS$ is greater than a right angle, points on $l$ with larger $x$-coordinate will have *smaller* $y$-coordinate. This means that both sides of (1) will be negative, since in each side the numerator and denominator will have the opposite sign.

**Problem 1.** Suppose $l$ is a line not parallel to the $y$-axis. Suppose $P_1 = P(x_1, y_1)$ and $P_2 = P(x_2, y_2)$ are two different points on $l$. Let $m = (y_2 - y_1) / (x_2 - x_1)$ and let $b = y_1 - m x_1$ . Show the following:

1. If $P = P(x, y)$ is any point on $l$, then $y = m x + b$.

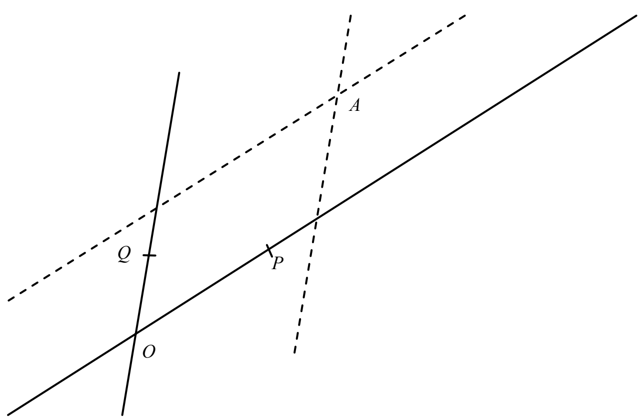
2. If $(x, y)$ is a solution to the equation $y = m x + b$, then $P(x, y)$ is on line $l$. (Or equivalently: if $P = P(x, y)$ is NOT on $l$, then $y \neq m x + b$.)

## Skew coordinates

A skew coordinate system in the plane is made in exactly the same way as a standard coordinate system except:

- the two axes do NOT have to be perpendicular;
- the unit intervals on the two axes $OP$ and $OQ$ do NOT have to have the same length.

In other words, we just omit the parts of the definition of standard system that were underlined above. Otherwise, the construction of a skew coordinate system is exactly the same as a standard system. Here is a diagram showing a skew coordinate system and the lines parallel to the axes used to determine the $x$- and $y$- coordinates of a point $A$.




**Problem 2.** Show that Fact 3 (above) remains true in a skew coordinate system. Hint. You need to imitate all the arguments used in the case of a standard system---that is, the arguments on pages 48-49, as well as the arguments I made for Fact 3. But some strategic alterations need to be made.

**Problem 3.** Is the analogue of Problem 1 true for skew coordinate systems?

**Problem 4. True or false?** Given any linear equation (in two variables), its graph *relative to a given skew coordinate system* is a line. **True or false?** Given any line and any skew coordinate system, there a linear equation whose graph *relative to that system* is the given line.