Math 4005 Geometry, Spring 2005
Exercises

The problems are ranked according to my estimate of their difficulty:

- [1] A routine application of fundamental ideas or concepts.
- [2] Requires careful thought and at least two drafts.
- [3] A challenging problem will probably require a period of intense work, a rest and a second try at a later time.
- [4] A deep and difficult problem that might require several days and numerous attempts.
- [5] A problem that I have not solved.

In many problems, you are asked to "demonstrate" your answer. What do I mean by demonstrate? I mean the same thing that Lincoln meant when he said "demonstrate".

> *In the course of my law reading I constantly came upon the word demonstrate. I thought, at first, that I understood its meaning, but soon became satisfied that I did not. I said to myself, What do I do when I demonstrate more than when I reason or prove? How does demonstration differ from any other proof? I consulted Webster's Dictionary. They told of certain proof, proof beyond the possibility of doubt; but I could form no idea of what sort of proof that was. I thought a great many things were proved beyond the possibility of doubt, without recourse to any such extraordinary process of reasoning as I understood demonstration to be. I consulted all the dictionaries and books of reference I could find, but with no better results. You might as well have defined blue to a blind man. At last I said, Lincoln, you never can make a lawyer if you do not understand what demonstrate means; and I left my situation in Springfield, went home to my father's house, and stayed there till I could give any proposition in the six books of Euclid at sight. I then found out what demonstrate means, and went back to my law studies.*
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> Abraham Lincoln quoted in *The Independent* for September 1, 1864

> *There are two ways of establishing a proposition. One is by trying to demonstrate it upon reason, and the other is, to show that great men in former times have thought so and so, and thus to pass it by the weight of pure authority. Now, if Judge Douglas will demonstrate somehow that this is popular sovereignty, the right of one man to make a slave of another, without any right in that other, or anyone else to object—demonstrate it as Euclid demonstrated propositions—there is no objection. But when he comes forward, seeking to carry a principle by bringing it to the authority of men who themselves utterly repudiate that principle, I ask that he shall not be permitted to do it.*
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> Abraham Lincoln, *Debate with Douglas, Columbus, Ohio, Sept. 1859*

Problem: [3] What does it mean to demonstrate?

CLASS of January 18, 2006.

1. Mathematical definitions

1.1. [1^+] Euclid defines a *point* as "that which has no part". Does this make the concept clear? How would you explain the meaning of the idea of a point to a 12-year-old? Would your explanation be a mathematical definition?

1.2. [1] Euclid defines a *line* as "breadthless length, and a *straight line* as "a line that lies evenly with the points on itself." As explanations, how are these? Are these statements adequate as mathematical definitions? Why or why not?

1.3. Moise (*Elementary Geometry from an Advanced Standpoint*, page 55) defines a *ray* as follows: "The ray $\overrightarrow{AB}$ is the set of all points C on the line $\overleftrightarrow{AB}$ such that A is *not* between C and B."

1.3.1. [1] Draw a picture of ray $\overrightarrow{AB}$ and describe in writing how and why the object you drew satisfies the definition.

1.3.2. [1] What definitions and concepts are presupposed in the statement of the definition of ray?

1.3.3. [1] Give a definition of *ray* that does not use the word "not" and uses no geometric concepts (such as "beyond" or "infinity") that do not already appear in Moise's definition.

1.4. [1] Define *angle*.

1.5. [2^+] What is a mathematical definition?

2. Lines

2.1. [2] Given n points in the plane no three of which lie on a line, how many distinct lines can be drawn through pairs of those points? Write down a formula in terms of n for the number of such lines.

2.2. Determine whether the statement is true or false and provide an explanation of your reasoning.

2.2.1. [1] Given *any* set of 3 points in the plane that do not all lie on one line, one always can draw 3 distinct lines each of which passes through exactly 2 of the points.

2.2.2. [2] Given *any* set of 4 points in the plane that do not all lie on one line, one always can draw 4 distinct lines each of which passes through exactly 2 of the points.

2.2.3. [2] Given *any* set of 4 points in the plane that do not all lie on one line, one always can draw 3 distinct lines each of which passes through exactly 2 of the points.

2.2.4. [3] Given *any* set of 5 points in the plane that do not all lie on one line, one always can draw 4 distinct lines each of which passes through exactly 2 of the points.

2.2.4. [4] Given *any* set of n points in the plane that do not all lie on one line, one always can draw $n-1$ distinct lines each of which passes through exactly 2 of the points.

2.3 [2] Euclid's second postulate is

To produce a finite straight line continuously in a straight line.

What is a "finite straight line"? What is the meaning of this postulate? Restate it using modern terminology.

3. Constructions

3.1 [2] Describe and illustrate and justify the steps used to draw the perpendicular bisector of a segment in a way that would be appropriate in a rigorous high school lesson.

3.2 [2] Same problem for the bisector of an angle.

3.3 [2] Same problem, where the task is to draw the line through point P that is perpendicular to line $\overleftrightarrow{AB}$. Note that P might—or might not—be on $\overleftrightarrow{AB}$. Your instructions should cover both cases. If necessary, you can make two different sets of instructions.

3.4 [2] Same problem, , where the task is to draw the line parallel to $\overleftrightarrow{AB}$ through point P, a point not on $\overleftrightarrow{AB}$.

3.5 [3^+] You have a rusty compass. It is opened to a span of 1 inch and it cannot be adjusted. You are given two points A and B that are less than two inches apart. You are allowed to use the rusty compass to draw circles around A and B. After this, you may also draw a circle of radius 1 around any point that occurs at the intersection of any two circles that you have already drawn. You may continue as long as you please, using the rusty compass to draw a circle around any point where two previously drawn circles cross one another. Question: Under what conditions on A and B can you construct a point C so that ABC is equilateral? Demonstrate your claim. (Remark: Certainly of A and B are one inch apart, you can succeed in locating C by drawing just two circles. The question is, are there *other* distances from A to B when the equilateral triangle can be drawn (possibly requiring more circles), and what are *all* such distances?

4. Problems from the book

1.3.6, 1.4.1, 1.4.2, 1.4.3, 1.4.4.