REVIEW for TEST II

Note to Students

The test on April 19 will be based closely on these review notes. These notes, in turn, are based directly on the notes given out previously for in-class work and homework. In some places, I have taken themes from our previous work and developed them a little further. The idea I have in mind is that you should explore somewhat beyond the minimum understandings that you'll be called upon to use on the test.

Remark on Friday, April 7, 5:40PM. I've been working hard, trying to meet my self-imposed deadline of making these note available today. Unfortunately, I find I still have more to do. I am going to post these now, even though they are not complete and finish them tomorrow. Thanks for your patience. —JJM

Functions

- Vocabulary: function, domain, codomain injective, surjective, bijective, inverse function.
- Notation: "$f : X \to Y$," means, "f is a function with domain X and codomain Y." $f(a)$ is the "value of f at a." (Here, a is in the domain of f and $f(a)$ is the corresponding element of the codomain.)
- Problems:
  1. If possible, give an example of a function f with domain $\{ a, b, c, d \}$ and codomain $\{ 1, 2, 3, 4 \}$ that satisfies the requirement, or explain why the requirement cannot be met: a) f is injective but not surjective, b) f is surjective but not injective, c) f is bijective.
  2. Suppose $f : \{ a, b, c, d \} \to \{ 1, 2, 3, 4 \}$ is the function that makes the following assignments: $f(a) = f(b) = f(c) = 1$ and $f(d) = 2$. What is the graph of f?
  3. In algebra, when we write something like $y = x^2 - x + 1$ we often say, "y is a function of x." Reconcile this with the definition of function that I gave: "A function from S to T is a rule that instructs us to select a certain element of T whenever we are given an element of S. The selection must depend only on the element of S given.
  4. What is a transformation of the plane?

Coordinate systems

- Topics and concepts
  - How to impose a coordinate system on the plane. Coordinate systems with perpendicular and non-perpendicular axes and possibly different units on the two axes.
  - Coordinates x and y are functions from the plane to $\mathbb{R}$. A coordinate system creates a correspondence between the points in the plane and the set $\mathbb{R}^2$ of ordered pairs of real numbers:

$$\text{point in the plane, } A \mapsto (x(A), y(A)), \text{ a pair of real numbers;}$$
$$\text{pair of real numbers, } (a, b) \mapsto P(a, b), \text{ a point in the plane.}$$

- ∘ Definition of the slope $m(A, B)$ between a pair of points (within a given coordinate system). Fact: $m(A, B) = m(B, C)$ iff A, B and C are collinear. Role of similar triangles in proving this.
- ∘ Given any equation $\mathcal{E}$ involving x and y, the collection of number pairs that make the equation true is the *graph* of the equation:

$$\text{graph of } \mathcal{E} := \text{set of all pairs } (x, y) \text{ such that } \mathcal{E}(x, y) \text{ is true.}$$

  If the pairs are viewed as locations in the plane determined by some coordinate system, then the graph of $\mathcal{E}$ is a figure of some sort. The same idea applies if $\mathcal{E}$ is any relation at all. For example, suppose $\mathcal{E}(x, y)$ is the statement $y > x$. Then, the graph of $\mathcal{E}$ is set of all pairs (x, y) such that $y > x$. Interpreted geometrically, this is the set of all points strictly above the diagonal line $y = x$.
- ∘ In a standard coordinate system (*i.e.*, perpendicular axes and the same standard unit of distance on both) the distance $\operatorname{dist}(A, B)$ between points A and B satisfies the following:

$$\big(\operatorname{dist}(A, B)\big)^2 = (x(B) - x(A))^2 + (y(B) - y(A))^2.$$

  (This is the Pythagorean Theorem.) Accordingly, the graph of the equation $r^2 = (a - x)^2 + (b - y)^2$, $r > 0$, is the set of points at distance r from $P(a, b)$.

- Problems:
  1. Given non-collinear points O, P and Q, suppose that a coordinate system is set up with O at $(0, 0)$, P at $(1, 0)$ and Q at $(0, 1)$. What are the coordinates of the following points: **a)** midpoint of OP, **b)** fourth vertex of parallelogram with vertices O, P and Q, **c)** midpoint of PQ, **d)** (hard) the point of intersection of lines ℓ and m, where ℓ is the line that joins P and the midpoint of OQ and m is the line that joins Q and the midpoint of OP *Answers*: **a)** $(\frac{1}{2}, 0)$, **b)** $(1, 1)$, **c)** $(\frac{1}{2}, \frac{1}{2})$ (*Proof.* Draw the segments joining the midpoints of each pair of opposite sides of the parallelogram in b). These segments meet at the common midpoint of the two diagonals.) **d)**$(\frac{1}{3}, \frac{1}{3})$ (*Proof.* Line ℓ contains points $(1, 0)$ and $(0, \frac{1}{2})$, so its equation is $2y = -x + 1$. Line m contains points $(0, 1)$ and $(\frac{1}{2}, 0)$, so its equation is $y = -2x + 1$. Using the second equation to eliminate y from the first, we get $2(-2x + 1) = -x + 1$, which simplifies to $x = \frac{1}{3}$. Using this for x in the first equation gives $y = \frac{1}{3}$.
  2. In a standard coordinate system, find the equation for the set points equidistant from two distinct points (a, b) and (c, d). *Solution*: The coordinates (x, y) of such a point must satisfy:

$$(x - a)^2 + (y - b)^2 = (x - c)^2 + (y - d)^2.$$

     Expanding and cancelling the squared terms that are repaeated on both sides, we get

$$-2ax + a^2 + -2by + b^2 = -2cx + c^2 + -2dy + d^2,$$

and simplifying, we get the linear equation:

$$2(c-a)x + 2(d-b)y + a^2 + b^2 - c^2 - d^2 = 0.$$

Note that $c-a$ and $d-b$ cannot both be zero, because we assumed that (a,b) and (c,d) were different.

Linear functions

- An expression of the form $ax+by$, where a and b are real numbers (not both 0) and x and y are variables, is said to be *homogeneous linear in x and y*. An expression of the form $ax+by+c$, where a and b are real numbers (not both 0), c is a nonzero real and x and y are variables, is said to be *affine linear in x and y*. An equation of the form
$$ax+by+c=0, \tag{$*$}$$
where at least one of the coefficients a or b is nonzero is said to be linear. Equations that are obviously equivalent to equations of this form, *e.g.*,
$$y=1,\ x=2 \text{ or } y=3x+2,$$
are also called linear. If $c=0$, we say the equation $(*)$ is homogeneous linear. Otherwise, we say it's affine linear.
- Let A be a point with coordinates $x_1 = x(A)$ and $y_1 = y(A)$ and let m be a real number. Then
$$\{\, P(x,y) \mid m = \frac{x-x_1}{y-y_1}\,\}$$
is a line.

Transformations

Complex numbers