Hints for homework assigned Sept. 29
Originally due Oct. 4; revised due date Oct. 6.

4.26 (Needed for **7.14**). Here, we have random variables

- X and Y independent, with $f_X(x) = \alpha e^{-\alpha x}$ and $f_Y(y) = \beta e^{-\beta y}$,
- $Z := \min\{X, Y\}$,
- W, with $W = 1$ iff $X \le Y$, and $W = 0$ iff $Y < X$.

We are asked to show that Z and W are independent. Let us calculate the joint distribution $F_{Z,W}(z,i) := P(Z \le z \;\&\; W = i)$ by integrating over appropriate regions in the plane:

$$\begin{aligned} P(Z \le z \;\&\; W = 1) &= P(X \le z \;\&\; X \le Y) \\ &= \int_0^z \int_x^\infty \alpha e^{-\alpha x} \beta e^{-\beta y}\, dy\, dx \\ &= \cdots \textit{ (You provide details.) } \cdots \\ &= \frac{\alpha}{\alpha+\beta}\Big(1 - e^{-(\alpha+\beta)z}\Big). \end{aligned}$$

Similarly

$$\begin{aligned} P(Z \le z \;\&\; W = 0) &= \cdots \\ &= \frac{\beta}{\alpha+\beta}\Big(1 - e^{-(\alpha+\beta)z}\Big). \end{aligned}$$

Also, show by integrating over the appropriate potions of the plane:

$$P(W = i) = \begin{cases} \frac{\beta}{\alpha+\beta}, & \text{if } i = 0; \\ \frac{\alpha}{\alpha+\beta}, & \text{if } i = 1. \end{cases}$$

Finally, calculate

$$\begin{aligned} P(Z \le z) &= 1 - P(X > z \;\&\; Y > z) \\ &= 1 - P(X > z) \cdot P(Y > z) \\ &= \cdots \\ &= 1 - e^{-(\alpha+\beta)z}. \end{aligned}$$

It is now easy to verify independence of Z and W.

7.14 We can apply **4.26** with $1/\lambda = \alpha$ and $1/\mu = \beta$. By independence, the joint distribution is the *pdf* f_Z times the *pmf* f_W, where

$$\begin{aligned} f_Z(z|\alpha,\beta) &= (\alpha+\beta)e^{-(\alpha+\beta)z}, \quad z \in (0,\infty), \\ f_W(w|\alpha,\beta) &= \frac{w\alpha + (1-w)\beta}{\alpha+\beta}, \quad w = 0, 1. \end{aligned}$$

7.19 Here, σ^2 is the same for all the ϵ_i. Note that $Y_i \sim \text{normal}(\beta x_i, \sigma^2)$, so

$$f_{Y_i}(y_i) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(y_i - \beta x_i)^2}{2\sigma^2}\right).$$

(a) Begin by writing $f_{\vec{Y}}(\vec{y} \mid \beta, \sigma^2)$, and then use the factorization theorem.
(b) Use the log likelihood. The MLE is a linear combination of Y_i, so the expected value is easy to find.
(c) See the hint for (b).