13. Suppose, to be specific, that in Problem 12, $\theta_0 = 1$, $n = 10$, and that $\alpha = .05$. In order to use the test, we must find the appropriate value of c.

- **a.** Show that the rejection region is of the form $\{\overline{X} \leq x_0\} \cup \{\overline{X} \geq x_1\}$, where x_0 and x_1 are determined by c.
- **b.** Explain why c should be chosen so that $P(\overline{X}\exp(-\overline{X}) \leq c) = .05$ when $\theta_0 = 1$.
- **c.** Explain why $\sum_{i=1}^{10} X_i$ and hence $\overline{X}$ follow gamma distributions when $\theta_0 = 1$. How could this knowledge be used to choose c?
- **d.** Suppose that you hadn't thought of the preceding fact. Explain how you could determine a good approximation to c by generating random numbers on a computer (simulation).

14. Suppose that under H_0, a measurement X is $N(0, \sigma^2)$, and that under H_1, X is $N(1, \sigma^2)$ and that the prior probability $P(H_0) = 2 \times P(H_1)$. As in Section 9.1, the hypothesis H_0 will be chosen if $P(H_0|x) > P(H_1|x)$. For $\sigma^2 = 0.1, 0.5, 1.0, 5.0$:

- **a.** For what values of X will H_0 be chosen?
- **b.** In the long run, what proportion of the time will H_0 be chosen if H_0 is true $\frac{2}{3}$ of the time?

15. Suppose that under H_0, a measurement X is $N(0, \sigma^2)$, and that under H_1, X is $N(1, \sigma^2)$ and that the prior probability $P(H_0) = P(H_1)$. For $\sigma = 1$ and $x \in [0, 3]$, plot and compare (1) the p-value for the test of H_0 and (2) $P(H_0|x)$. Can the p-value be interpreted as the probability that H_0 is true? Choose another value of σ and repeat.

16. In the previous problem, with $\sigma = 1$, what is the probability that the p-value is less than 0.05 if H_0 is true? What is the probability if H_1 is true?

17. Let $X \sim N(0, \sigma^2)$, and consider testing $H_0 : \sigma = \sigma_0$ versus $H_A : \sigma = \sigma_1$, where $\sigma_1 > \sigma_0$. The values σ_0 and σ_1 are fixed.

- **a.** What is the likelihood ratio as a function of x? What values favor H_0? What is the rejection region of a level α test?
- **b.** For a sample, $X_1, X_2, \ldots, X_n$ distributed as above, repeat the previous question.
- **c.** Is the test in the previous question uniformly most powerful for testing $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma > \sigma_0$?

18. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables from a double exponential distribution with density $f(x) = \frac{1}{2}\lambda \exp(-\lambda|x|)$. Derive a likelihood ratio test of the hypothesis $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda = \lambda_1$, where λ_0 and $\lambda_1 > \lambda_0$ are specified numbers. Is the test uniformly most powerful against the alternative $H_1 : \lambda > \lambda_0$?