M4056 Lecture Notes. **September 10, 2010**

Transformations

These notes are intended to clarify a point in the last lecture.

Suppose we have two copies of $\mathbb{R}^n$, one with coordinates $x_1, \ldots, x_n$ and one with coordinates $u_1, \ldots, u_n$. Suppose also that we have a pair of differentiable functions g and h that establish a bijection between the two copies of $\mathbb{R}^n$, with

$$\vec{u} = g(\vec{x}) \text{ and } \vec{x} = h(\vec{u}).$$

This means that we can write each x_i as a function of the u_js and each u_j as a function of the x_is.

Example. $\vec{u}^{\,T} = A \cdot \vec{x}^{\,T}$ for some invertible matrix A (so $\vec{x}^{\,T} = A^{-1} \cdot \vec{u}^{\,T}$).

Now suppose $\vec{X}$ is a random variable with *pdf* $f_{\vec{X}}(\vec{x})$. Then $\vec{U} := g(\vec{X})$ is also a random variable, and it has a *pdf* $f_{\vec{U}}(\vec{u})$. What is the relationship between the two *pdf*s?

If E is a subset of $\mathbb{R}^n$, then the change of variables theorem says:

$$\int_{h(W)} f_{\vec{X}}(\vec{x}) d\vec{x} = \int_W f_{\vec{X}}(h(\vec{u}))|h'(\vec{u})| d\vec{u}.$$

Now,

$$\int_{h(W)} f_{\vec{X}}(\vec{x}) d\vec{x} = P(\vec{X} \in h(W)) = P(\vec{g}(X) \in gh(W)) = P(\vec{U} \in W)$$

Thus

$$P(\vec{U} \in W) = \int_W f_{\vec{X}}(h(\vec{u}))|h'(\vec{u})| d\vec{u}.$$

Since this is true for all W,

$$f_{\vec{U}}(\vec{u}) = f_{\vec{X}}(h(\vec{u}))|h'(\vec{u})|.$$

See (4.3.2) on page 158 and the discussion around it. See also Example 4.6.13.