M4056 Lecture Notes. **September 15, 2010**

Some important distributions

Our goal is to investigate the properties of the density function of S_n^2 that we discovered last time. This is the so-called chi squared density with $n-1$ degrees of freedom. It is best understood in the context of a class of probability densities called the *gamma densities*. To define them, we need to introduce (or renew our acquaintance with) an interesting function. (See page 99.)

The *gamma function*, Γ, is defined

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt.$$

Homework 1.

a) Show that $\Gamma(1) = 1$.
b) Using integration by parts, show $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.
c) Using a) and b), show that $\Gamma(n) = (n-1)!$.

The *gamma density* with parameters $\alpha > 0$ and $\beta = 1$ is defined to be:

$$g(t|\alpha, 1) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}e^{-t}\,, \quad 0 < t < \infty.$$

Homework 2.

a) Verify that $g(t|\alpha, 1)$ is a probability density.
b) Suppose T is has a gamma distribution with parameters α and 1. Show that $X = \beta T$ has the following distribution. (See page 99, (3.3.6).)

$$g(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}\,, \quad 0 < x < \infty.$$

We shall calculate the moment generating function of $g(x|\alpha, \beta)$. (See 2.3.8.)

$$M_X(t) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty e^{tx}x^{\alpha-1}e^{-x/\beta}.$$

The integrand can be written in the form $x^{\alpha-1}e^{-x(\frac{1}{\beta}-t)}$, or in the form $x^{\alpha-1}e^{-x/B(t)}$, with $B(t) = \frac{\beta}{1-\beta t}$. Now, we know (from Homework 2.b) that

$$1 = \frac{1}{\Gamma(\alpha)B(t)^\alpha}\int_0^\infty e^{tx}x^{\alpha-1}e^{-x/B(t)}dx.$$

So,

$$M_X(t) = \frac{\Gamma(\alpha)B(t)^\alpha}{\Gamma(\alpha)\beta^\alpha} = (1-\beta t)^{-\alpha}\,, \quad t < 1/\beta.$$

Now we return to our main theme. Suppose X is $n(0,1)$. We calculate the *cdf* of X^2.

$$\begin{aligned} F_{X^2}(x) &= P(X^2 \le x) \\ &= P(-\sqrt{x} \le X \le \sqrt{x}) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{x}}^{\sqrt{x}} e^{-t^2/2} dt \\ &= \frac{2}{\sqrt{2\pi}} \int_{0}^{\sqrt{x}} e^{-t^2/2} dt. \end{aligned}$$

To find the *pdf*, we differentiate:

$$\begin{aligned} f_{X^2}(x) &= \frac{d}{dx} F_{X^2}(x) \\ &= \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/2} \end{aligned}$$

This is called the *chi squared density with one degree of freedom*. (The pronunciation is "KAI".) It is customary to use χ_1^2 to denote a random variable with this distribution. We recognize the kernel (i.e., non-constant part) of this distribution as the kernel of the gamma(1/2,2) density,

$$\frac{1}{\Gamma(1/2)2^{1/2}} x^{-1/2} e^{-x/2}.$$

Two densities with the same kernel must be equal. Thus we obtain the useful fact that

$$\Gamma(1/2) = \sqrt{\pi}.$$

By the discussion of the gamma density, the *mgf* of χ_1^2 is

$$M_{\chi_1^2}(t) = (1-2t)^{-1/2}.$$

We can use this to determine the *mgf* of the sum $X_1 + \cdots + X_n$, where X_i *iid* χ_1^2. It is $(1-2t)^{-n/2}$, and this is the *mgf* of the gamma density $g(x|n/2,2)$. We call a random variable with this density *chi squared with n degrees of freedom*. Such a variable is customarily denoted χ_n^2.

Summary: If $Z_1, \ldots, Z_n$ are *iid* $n(0,1)$, then $Z_1^2 + \cdots + Z_n^2$ is χ_n^2. The *pdf* of this distribution is gamma with parameters $\alpha = n/2$ and $\beta = 2$.