M4056 Lecture Notes. **September 17, 2010**

Some important distributions, continued. Student's t_p and Snedecor's $F_{p,q}$

Suppose we take samples of size n from a $n(\mu, \sigma^2)$ population. In this case, as we have seen,

$$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim n(0,1).$$

This serves us well if our intention is to estimate an unknown μ *and we know* σ. For example, suppose a laboratory instrument measures lengths with a standard error of 0.02 millimeters. (In other words, if the same object is measured repeatedly, the readings are normally distributed about the true length, with $\sigma = 0.02$.) To increase accuracy, we decide to measure each object 25 times and use the average of the 25 readings as our estimate of true length.

Homework 1. If 1000 objects are measured in this fashion (requiring 25,000 uses of the instrument), how many are likely to have a true length that is no more than 0.01 millimeters from the estimate?

But it is unusual to know σ. (For example, the measuring instrument might have a standard error that varied with the operator and the weather.) As long as the errors all have the same normal distribution, $\overline{X}$ will be the best estimate of the true value μ. (We will examine what "best" means later.) But typically we want more. If we base each of many estimates on n measurements, can we place bounds on the size of the errors that we will commit with given frequency? We saw that this was possible above, if σ was known. But what if it's not? This problem was addressed by William Sealy Gosset a mathematician who worked for the Dublin brewery of Arthur Guinness & Son. (See the Wikipedia article on Gosset for interesting details.)

We have seen above that S^2 is a good estimate of σ^2. Therefore, we might expect $\frac{\overline{X}-\mu}{S/\sqrt{n}}$ to be useful. Now,

$$\frac{\overline{X} - \mu}{S/\sqrt{n}} = \frac{(\overline{X} - \mu)/(\sigma/\sqrt{n})}{\sqrt{S^2/\sigma^2}} \sim \frac{U}{\sqrt{\chi_p^2/p}},$$

where U is $n(0,1)$ and $p = n - 1$. Let T stand for this variable. Then

$$f_T(t) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \frac{1}{(p\pi)^{1/2}} \frac{1}{(1 + t^2/p)^{(p+1)/2}} \qquad (*)$$

Homework 2.

a) Study Theorem 5.2.9, page 215.
b) Do Problem 5.6 on page 256.
c) Study the derivation of $(*)$, on the bottom of page 223.

Homework 3. Study the definition of $F_{p,q}$ (5.3.5–5.3.7, page 224)