M4056 Confidence Intervals **November 15, 2010**

Let θ be a parameter of a probability distribution. A *confidence interval for* θ is a random interval, calculated from a sample, that contains θ with some specified probability. (A random interval is an interval $[L(\vec{X}), U(\vec{X})]$, where L and U are statistics.)

Example 1. Consider a sample X of size 1 an from a normal population with (unknown) mean θ and (known) variance σ^2. Let us find a 95% confidence interval for θ. This is easily done by normalizing X.

$$\begin{aligned}
\theta \in [X - r, X + r] &\Leftrightarrow X \in [\theta - r, \theta + r] \\
&\Leftrightarrow X - \theta \in [-r, r] \\
&\Leftrightarrow \frac{X - \theta}{\sigma} \in [-r/\sigma, r/\sigma] \\
&\Leftrightarrow Z \in [-r/\sigma, r/\sigma],\ Z \text{ standard normal.}
\end{aligned}$$

Now, $P(Z \in [-2, 2]) = 0.9545\ldots$, so if we set $r = 2\sigma$ we get the desired interval.

Example 1, continued. There is a direct connection to hypothesis testing. Let H_0 be the hypothesis that $\theta = \theta_0$. Suppose we wish to test this against the alternative $H_1 : \theta \neq \theta_0$, and we desire a test of level 0.05. If we reject H_0 when $\theta_0 \notin [X - 2\sigma, X + 2\sigma]$, our test will have size $1 - 0.954\ldots = 0.045\ldots$.

Example 2. The previous example shows how to deal with samples $\vec{X}$ of size n from a normal population with (unknown) mean θ and (known) variance σ^2, since in this case the distribution of the sample mean is normal with variance σ^2/n.

Example 3. Consider a sample $\vec{X}$ of size n an from a normal population with (unknown) mean μ and *unknown variance*. Let $\overline{X}$ be the sample mean and let S^2 be the sample variance. Lacking a knowledge of σ, cannot write an expression for the distribution of $\overline{X}$. However, we showed previously that

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

has a t distribution with $n - 1$ degrees of freedom. Recall that this is the t distribution with k degrees of freedom is $Z/\sqrt{\chi_k^2/k}$, where Z is standard normal.

The t distribution is symmetric, and resembles the normal distribution but it is somewhat flatter. We can use it in a manner perfectly analogous to the way we used the normal in Example 1. If we want the probability that $\mu \in [L(\vec{X}), U(\vec{X})$ to be q, we pick b so that $P(T \in [-b, b]) = q$. (Use the computer or a table of the t-distribution to do so.) Now

$$\begin{aligned}
T \in [-b, b] &\Leftrightarrow \overline{X} - \mu \in [-bS/\sqrt{n}, bS/\sqrt{n}] \\
&\Leftrightarrow \mu - \overline{X} \in [-bS/\sqrt{n}, bS/\sqrt{n}] \\
&\Leftrightarrow \mu \in [\overline{X} - bS/\sqrt{n}, \overline{X} + bS/\sqrt{n}].
\end{aligned}$$

Example 3, continued. Suppose we wish to test the hypothesis $H_0 : \mu = \mu_0$ against the alternative $H_1 : \mu \neq \mu_0$, and we want a level of significance α. To achieve this goal, select b so that $P(T \in [-b, b]) = 1 - \alpha$. Reject H_0 when $\mu_0 \notin [\overline{X} - bS/\sqrt{n}, \overline{X} + bS/\sqrt{n}]$. By the equivalences above, the probability of rejecting H_0 when H_0 is true is α. The required values for b can be found from tables of the t-distribution, or from an appropriate computer program. If $n = 5$ and $\alpha = 0.05$, then b is about 2.777. If $n = 12$ and $\alpha = 0.05$, then b is about 2.201. If $n = 50$ and $\alpha = 0.05$, then b is about 2.011.

M4056 Comparing Independent Samples **November 17, 2010**

Suppose $\vec{X} = (X_1, \ldots, X_n)$ is i.i.d. normal(μ_X, σ^2). Let $\vec{Y} = (Y_1, \ldots, Y_n)$ be another sample that is i.i.d. normal(μ_Y, σ^2). Assume that $\vec{X}$ and $\vec{Y}$ are independent of one another. We wish to test $H_0 : \mu_X = \mu_Y$ versus $H_1 : \mu_X \neq \mu_Y$.

Consider the statistic $\overline{X} - \overline{Y}$. This is normal$(\mu_X - \mu_Y, \sigma^2/n + \sigma^2/m)$. If we knew σ^2, then we could deal with this as in Example 2, above.

On the other hand, if σ^2 is unknown—but the same for both X and Y—we may use an approach similar to Example 3. Let $S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ and $S_Y^2 = \frac{1}{m-1}\sum_{i=1}^{m}(Y_i - \overline{Y})^2$. The test statistic will involve the "pooled variance"

$$S_p^2 := \frac{(n-1)S_X^2 + (m-1)S_Y^2}{m+n-2}.$$

Theorem. *The statistic*

$$t = \frac{(\overline{X} - \overline{Y}) - (\mu_X - \mu_Y)}{S_p\sqrt{1/n + 1/m}}$$

is a t distribution with $m + n - 2$ degrees of freedom.

Proof. $(n-1)S_X^2/\sigma^2 \sim \chi^2_{n-1}$ and $(m-1)S_Y^2/\sigma^2 \sim \chi^2_{m-1}$, so $S_p^2/\sigma^2 \sim \chi^2_{m+n-2}/(m+n-2)$.

$$t = \frac{(\overline{X} - \overline{Y}) - (\mu_X - \mu_Y)}{\sigma\sqrt{1/n + 1/m}} \frac{\sigma}{S_p} \sim Z/\sqrt{\chi^2_{m+n-2}/(m+n-2)}.$$