Problem 4.

a) $\Lambda(x_i) = \frac{P(X=x_i|H_0)}{P(X=x_i|H_1)}$. Therefore,

$$\begin{array}{rcccc} \Lambda(x_1) & = & 0.2/0.1 & = & 2 \\ \Lambda(x_2) & = & 0.3/0.4 & = & 3/4 \\ \Lambda(x_3) & = & 0.3/0.1 & = & 3 \\ \Lambda(x_4) & = & 0.2/0.4 & = & 1/2 \end{array}$$

b) $\Lambda(x_4) < \Lambda(x_2) < \Lambda(x_1) < \Lambda(x_3)$. $P(\textit{reject } H_0 \mid H_0) = P(\text{Type I error}) = 0.2$ if the rejection region is $\{x_4\}$. $P(\textit{reject } H_0 \mid H_0) = P(\text{Type I error}) = 0.5$ if the rejection region is $\{x_4, x_2\}$.

c) This part of the exercise refers to the Bayesian paradigm. In the Bayesian paradigm, the competing hypotheses are assigned probabilities—the so-called *prior probabilities*—before testing. After the test these probabilities are revised using the likelihood ratio (as we illustrate in the next part) to give the so-called *posterior probabilities*. The *favored hypothesis* is the hypothesis with the greater posterior probability. If the priors are equal, as in this part of this problem, the favored hypothesis is simply the more likely one. Thus, H_0 is favored if $X = x_1$ or $X = x_3$.

d) In the Bayes paradigm, we assume that $P(H_0)$ and $P(H_1)$ are positive numbers that add to 1. We then apply the following fact about conditional probability:

$$P(H_i \mid A) \cdot P(A) = P(H_i \,\&\, A) = P(A \mid H_i) \cdot P(H_i).$$

This gives us

$$\frac{P(H_0 \mid A)}{P(H_1 \mid A)} = \frac{P(A \mid H_0)}{P(A \mid H_1)} \cdot \frac{P(H_0)}{P(H_1)}.$$

Thus, if $P(H_0) = a$, then $P(H_1) = 1 - a$, and

$$\frac{P(H_0 \mid X = x)}{P(H_1 \mid X = x)} = \Lambda(x) \cdot \frac{a}{1-a}.$$

Now,

$$\begin{aligned} H_0 \text{ is favored } &\Leftrightarrow\ 1 \le \frac{P(H_0 \mid X = x)}{P(H_1 \mid X = x)} \\ &\Leftrightarrow\ 1 - a \le \Lambda(x) \cdot a \\ &\Leftrightarrow\ (1 + \Lambda(x))^{-1} \le a. \end{aligned}$$

This means that given $X = x_i$, H_0 is favored if a, the prior probability of H_0, exceeds the numbers indicated in the following table:

$$\begin{array}{rccccc} x & : & x_3 & x_1 & x_2 & x_4 \\ (1 + \Lambda(x_2))^{-1} & : & 1/4 & 1/3 & 4/7 & 2/3 \end{array} .$$

Problem 5.

a) False. The significance level is an upper bound for the probability of rejecting the null hypothesis when it is true.

b) False. As the significance level decreases, stronger evidence for rejecting the null is demanded. This would lessen the power.

c) False. In the frequentist paradigm, the null hypothesis is either true or false; it does not have a probability. In the Bayesian paradigm, we do not reject a hypothesis; we modify the probability we attach to it.

d) False. The probability that the null is falsely rejected is the size of a test.

e) False. A Type I Error occurs when the test statistic is in the rejection region but H_0 is true.

f) False. This may be the case in applications, but it would depend on how the test was used. It does not depend on the test.

g) False. The power of a test is a function of the parameter (via the distribution determined by the parameter).

h) True. The likelihood ratio is a function of the sample, and the sample is a random variable. So, the likelihood ratio is, too.

Problem 7.

Let $W = \sum_{i=1}^{n} X_i$. If the X_i are independent and Poisson(λ), then W is Poisson($n\lambda$). Thus,

$$f(w \mid \lambda) = e^{-n\lambda}\frac{(n\lambda)^w}{w!}, \text{ and}$$

$$\Lambda(w) = \frac{e^{-n\lambda_0}(n\lambda_0)^w}{e^{-n\lambda_1}(n\lambda_1)^w} = e^{-n(\lambda_0-\lambda_1)}\left(\frac{\lambda_0}{\lambda_1}\right)^w.$$

Since $\lambda_0 < \lambda_1$, this is a decreasing function of w. Therefore, the rejection region will be of the form $\{\, w \mid w \geq k \,\}$, where k is chosen large enough to achieve the desired significance level. Indeed, set k so that $\sum_{w=k}^{\infty} e^{-n\lambda_0}\frac{(n\lambda_0)^w}{w!} < \alpha$.

Problem 8.

Test T with rejection region R is uniformly most powerful in a class $\mathcal{C}$ of tests if, given any other test T' in $\mathcal{C}$,

$$\beta(\theta) \geq \beta'(\theta)\,, \text{ for all } \theta \in \Theta_1.$$

Let us choose a rejection region $R_k = \{\, w \mid w \geq k \,\}$ as in Problem 7, thus defining a test T_k. Then by the Neyman-Pearson Lemma, for any $\lambda_1 > \lambda_0$ and any test T' of the same significance level as T_k

$$\beta_k(\lambda_1) = P(w \in R_k \mid \lambda_1) > P(w \in R' \mid \lambda_1) = \beta'(\lambda_1).$$

This shows that T_k is uniformly most powerful for the alternatives $H_0 : \lambda = \lambda_0$ versus $H_0 : \lambda > \lambda_0$.

Problem 10.

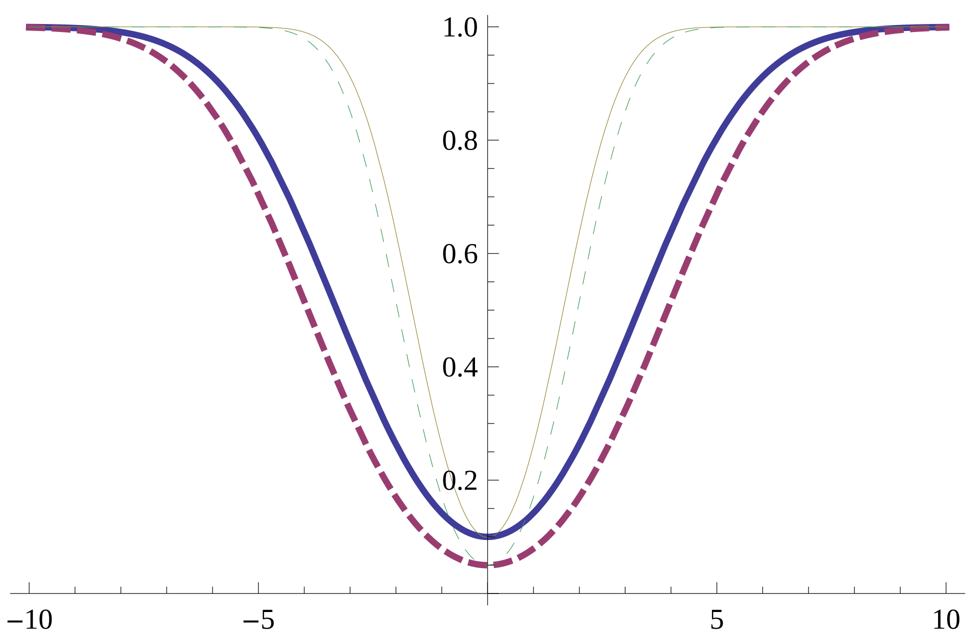
Suppose $T = T(\vec{X})$ is sufficient for $\vec{X}$. Then by the factorization theorem, $f(\vec{x} \mid \theta) = g(T(\vec{x}) \mid \theta)h(\vec{x})$, for some g and t. Thus,

$$\Lambda(\vec{x}) := \frac{f(\vec{x} \mid \theta_0)}{f(\vec{x} \mid \theta_1)} = \frac{g(T(\vec{x}) \mid \theta_0)h(\vec{x})}{g(T(\vec{x}) \mid \theta_1)h(\vec{x})} = \frac{g(T(\vec{x}) \mid \theta_0)}{g(T(\vec{x}) \mid \theta_1)}.$$

If the distribution of T under the null hypothesis is known, then to define a test of significance level α, we choose a rejection region R such that $P(T \in R \mid H_0) < \alpha$. With no further information, this is all that can be said.

Problem 11.

If $n = 25$, $\overline{X}$ is normal$(\mu, \sigma^2 = 4)$. At significance level 0.10, the rejection region is the complement of $[-3.29, 3.29]$. At significance level 0.05, the rejection region is the complement of $[-3.92, 3.92]$. If $n = 100$, $\overline{X}$ is normal$(\mu, \sigma^2 = 1)$. At significance level 0.10, the rejection region is the complement of $[-1.65, 1.65]$. At significance level 0.05, the rejection region is the complement of $[-1.96, 1.96]$. The graphs of the power function are shown below. Key: $n = 25, \alpha = 0.10$ (thick), $n = 25, \alpha = 0.05$ (thick, dashed), $n = 100, \alpha = 0.10$ (thin), $n = 100, \alpha = 0.05$ (thin, dashed). We decrease power by decreasing α. We increase power by increasing n.



Problem 12.

Here, $f(x \mid \theta) = \theta e^{-\theta x}$, so $f(\vec{x} \mid \theta) = \theta^n e^{-\theta(x_1+\cdots+x_n)} = \left(\theta e^{-\theta \overline{x}}\right)^n = f(\overline{x} \mid \theta)$. The log likelihood function is $\ell(\theta) = n(\ln\theta - \overline{x}\theta)$. Since $\frac{d\ell}{d\theta} = n(1/\theta - \overline{x})$, we see that the MLE of θ is $\hat{\theta} = 1/\overline{x}$. Then, the likelihood ratio test for $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is of the form (see bottom of page 375):

$$\text{reject } H_0 \text{ if } : \quad \lambda(\overline{x}) = \frac{f(\overline{x} \mid \theta_0)}{f(\overline{x} \mid \hat{\theta})} < k.$$

Now,

$$\frac{f(\overline{x} \mid \theta_0)}{f(\overline{x} \mid \hat{\theta})} = \frac{\left(\theta_0 e^{-\theta_0 \overline{x}}\right)^n}{\left(\hat{\theta} e^{-\hat{\theta}\overline{x}}\right)^n} = \left(\frac{\theta_0\, e^{-\theta_0 \overline{x}}}{(1/\overline{x})e^{-1}}\right)^n = \left(e\, \overline{x}\, \theta_0\, e^{-\theta_0 \overline{x}}\right)^n.$$

Thus, the test is of the form:

$$\text{reject } H_0 \text{ if } : \quad \overline{x}\, e^{-\theta_0 \overline{x}} < \frac{k^{1/n}}{e\, \theta_0}.$$