M4056 Home Work Answers **December 1, 2010**

Problem 12 (recalled)

In this problem, we are given $f(x \mid \theta) = \theta e^{-\theta x}$. If $\vec{X} = (X_1, \ldots, X_n)$ is an i.i.d. sample, then $f(\vec{x} \mid \theta) = \theta^n e^{-\theta(x_1+\cdots+x_n)} = \left(\theta e^{-\theta \overline{x}}\right)^n = f(\overline{x} \mid \theta)$.

The log likelihood function is $\ell(\theta) = n(\ln\theta - \overline{x}\theta)$. Since $\frac{d\ell}{d\theta} = n(1/\theta - \overline{x})$, we see that the MLE of θ is $\hat{\theta} := 1/\overline{x}$.

The likelihood ratio test for $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ is of the form (see bottom of page 375):

$$\text{reject } H_0 \text{ if :} \quad \lambda(\overline{x}) = \frac{f(\overline{x} \mid \theta_0)}{f(\overline{x} \mid \hat{\theta})} \leq k.$$

Now,

$$\frac{f(\overline{x} \mid \theta_0)}{f(\overline{x} \mid \hat{\theta})} = \frac{\left(\theta_0 e^{-\theta_0 \overline{x}}\right)^n}{\left(\hat{\theta} e^{-\hat{\theta}\overline{x}}\right)^n} = \left(\frac{\theta_0\, e^{-\theta_0\overline{x}}}{(1/\overline{x})e^{-1}}\right)^n = \left(e\,\overline{x}\,\theta_0\, e^{-\theta_0\overline{x}}\right)^n.$$

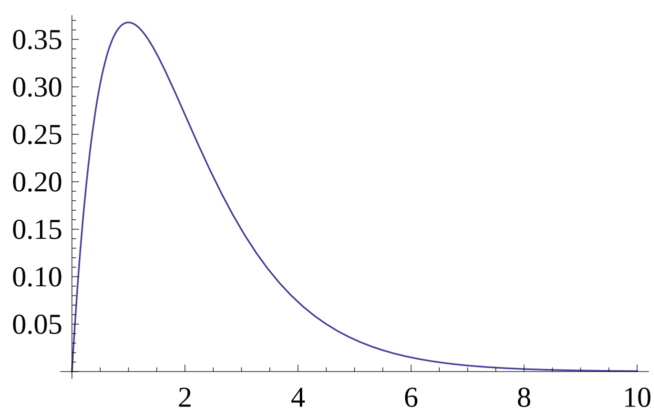
Thus, the test is of the form:

$$\text{reject } H_0 \text{ if :} \quad \overline{x}\, e^{-\theta_0\overline{x}} \leq c, \quad \text{where } c = \frac{k^{1/n}}{e\,\theta_0}.$$

Problem 13

Continuing Problem 12, suppose $\theta_0 = 1$, $n = 10$ and $\alpha = .05$. We seek the corresponding value of c to define the test.

a. A graph of the function $y = x\, e^{-x}$ appears below.



This makes it clear that $\{\, x \mid x\, e^{-x} \leq c \,\}$ is a union of two intervals: $[0, x_0] \cup [x_1, \infty)$, where x_0 and x_1 are determined by c. (They are the solutions to $x = ce^x$.)

b. We want to choose c so that $P(\overline{X}\, e^{\overline{X}} \leq c) = .05$ because $P(\overline{X}\, e^{\overline{X}} \leq c)$ is the probability of rejecting the null hypothesis when it is true, and we want this probability to be .05.

c. It is well-known that a sum of exponential random variables is gamma and that the mean of a sample from a gamma distribution is gamma. This can be demonstrated with moment generating functions. Indeed, $\theta\, e^{-\theta x}$ is gamma($\alpha = 1$, $\lambda = \theta$). The mgf of gamma(α, λ) is $\left(\frac{\lambda}{\lambda - t}\right)^{\alpha}$. Thus, the pdf of $10\overline{X}$ is gamma($\alpha = 10$, $\lambda = 1$) and the pdf of $\overline{X}$ is gamma($\alpha = 10$, $\lambda = 10$). (In *Mathematica* and many other places, gamma distributions are specified using the parameters α and β, where $\beta = \lambda^{-1}$.)

d. The *Mathematica* program

```
RandomReal[ExponentialDistribution[1], 10]
```

produces a random sample of size 10 from a population modeled by the exponential distribution with pdf $f(x) = e^{-x}$. The following produces the value of the mean of a random sample of size 10:

```
(1/10)Plus@@RandomReal[ExponentialDistribution[1], 10],
```

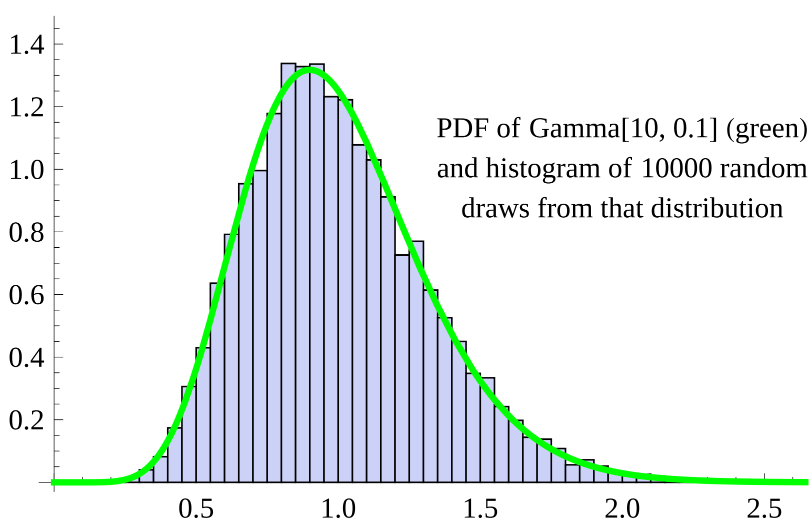
and the following produces 10,000 such means and sorts them by size:

```
Sort@Table[(1/10)Plus@@RandomReal[ExponentialDistribution[1], 10], 10000].
```

By our previous comments, this is the same as:

```
Sort@Table[RandomReal[GammaDistribution[10, 1/10], 10000].
```

The smallest 2.5% of the this collection have values less than or equal to the 250^{th} element of this list, which is 0.484609 (or on another try, 0.475156). The value 250 places from the end of the list is: 1.71341 (or on another try, 1.71225). Thus, for x_0, we may choose .47 and for x_1 take 1.72, and the test will have a significance level very close to the desired $\alpha = .05$.



Problem 17

In this problem, we are asked to examine $X \sim \text{normal}(0, \sigma^2)$ and develop the likelihood ratio test for $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma = \sigma_1$, where $\sigma_0 < \sigma_1$ are fixed.

a) First, we do this using a single sampled value of X. The likelihood ratio is:

$$\Lambda(x) = \frac{f(x|H_0)}{f(x|H_1)} = \frac{\sigma_1}{\sigma_0} \exp\Big(\frac{-x^2}{2\,\tau^2}\Big), \quad \tau^2 = \frac{\sigma_0^2 \sigma_1^2}{\sigma_1^2 - \sigma_0^2}.$$

Thus, the rejection region will be a pair of intervals $(-\infty, -r] \cup [r, \infty)$. To achieve a significance level of α, we simply choose r so that $\frac{1}{\sigma_0\sqrt{2\pi}} \int_{-r}^{r} e^{-x^2/2\sigma_0^2} = 1 - \alpha$.

b) If we have the data from an i.i.d. sample of size n, then the likelihood ratio is the product:

$$\Lambda(\vec{x}) = \frac{f(\vec{x}|H_0)}{f(\vec{x}|H_1)} = \prod_{i=1}^{n} \frac{\sigma_1}{\sigma_0} \exp\Big(\frac{-x_i^2}{2\,\tau^2}\Big) = \Big(\frac{\sigma_1}{\sigma_0}\Big)^n \exp\Big(\frac{-1}{2\,\tau^2} \sum_{i=1}^{n} x_i^2\Big).$$

This is small when $\sum_{i=1}^{n} x_i^2$ is large, so the rejection region is the exterior of an n-sphere in $\mathbb{R}^n$. Let $SS(n)$ be the statistic $\sum_{i=1}^{n} X_i^2$. We reject the null when $SS(n)$ is large; we choose how large based on the desired significance. Under the null hypothesis, $SS(n)/\sigma_0^2$ is χ^2_{n-1}, so a chisquare table can be used.

c) As discussed this previously in Problem 8. If we test $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma > \sigma_0$, our rejection region will be an interval for $SS(n)$ of the form $[r, \infty)$, where r is determined by the significance level. By Neyman-Pearson, such a test is most powerful for any specified alternative, so it's uniformly most powerful.

Problem 19

In this problem, we have two pdfs on $[0, 1]$: $f(x|0) = 2x$ and $f(x|1) = 3x^2$. We wish to test which distribution X obeys. The likelihood ratio is:

$$\Lambda(x) = \frac{2x}{3x^2} = \frac{2}{3x}.$$

a) $H_0 : f(x) = f(x|0)$ is favored if $x < \frac{2}{3}$, because then $\Lambda(x) > 1$.

b) The likelihood ratio test rejects H_0 if $2/(3x) \le c$, for some c. Therefore, the rejection region is an interval of the form $[r, 1]$.

c) To achieve level α, we must choose r so that $P(\mathit{reject} \,|\, H_0) = \alpha$, i.e., $\int_r^1 2x\, dx = \alpha$, i.e., $1 - r^2 = \alpha$, i.e., $r = \sqrt{1 - \alpha}$.

d) The power of the test with rejection region $[r, 1]$ is $P(\mathit{reject} \,|\, H_1) = \int_r^1 3x^2\, dx = 1 - r^3$.