M4056 Quiz 6, Oct. 1, 2010 Name______________________

Recall that the Poisson distribution with parameter $\lambda > 0$ is the discrete distribution with *pmf*

$$f(x|\lambda) = e^{-\lambda} \cdot \frac{\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

Suppose a sample $\vec{X} = (X_1, \ldots, X_n)$ is drawn from a Poisson distribution with unknown λ. Let $W = \sum_{i=1}^n X_i$ be the sample sum.

1. Write the *pmf* of the sample sum.
SOLUTION:
$$f_W(w|\lambda) = e^{-n\lambda} \cdot \frac{(n\lambda)^w}{w!}.$$

2. Write the likelihood function of λ, given that the sample is $(2, 3, 1, 2)$.

3. What is the MLE of λ in this case?

4. Show that in general, the MLE of λ is the sample mean.
SOLUTION:
$$L(\lambda|\vec{x}) = \prod_{i=1}^n e^{-\lambda} \cdot \frac{\lambda^{x_i}}{x_i!}$$
$$\ln L(\lambda|\vec{x}) = -n\lambda + \ln \lambda \sum_{i=1}^n x_i - \sum \ln(x_i!)$$
$$\frac{d}{d\lambda} \ln L(\lambda|\vec{x}) = -n + \frac{w}{\lambda}, \quad \text{with } w = \textstyle\sum_{i=1}^n x_i$$
Thus, as a function of λ, $L(\lambda|\vec{x})$ is increasing on $(0, w/n)$ and decreasing on $(w/n, \infty)$. So, the maximum is at w/n.

Comments (post-quiz).

The first problem is not needed to do the other ones. In fact, the solution of the first problem is dependent on techniques from Chapter 5. You might recall that the sum of several Poisson variables is Poisson, with parameter equal to the sum of the parameters of the summands. If you forgot (or never knew) this, you could use moment generating functions to rediscover and prove it. Or, you could use the chocolate chip cookie model: the Poisson gives the distribution of chips in cookies if the dough—enough for infinitely many cookies—contains enough chips to average λ per cookie. Accordingly, if we have enough dark chocolate chips for λ_1 per cookie and enough white chocolate chips for λ_2 per cookie, then we'll have on average $\lambda_1 + \lambda_2$ chips per cookie, and if there are n kinds of chips and enough of each kind for λ of that kind per cookie, the there will be enough for $n\lambda$ chips in all, per cookie. Hence the answer appearing above.

The MLE in problem 4 is the sample mean, $\overline{X} = W/n$. We will now go beyond the quiz to anticipate the work we'll do next week. The distribution of the MLE can be deduced from the distribution of the sample sum:

$$P(\overline{X} = t|\lambda) = P(W = nt|\lambda) = f_W(nt|\lambda).$$

We could write out the distribution of $\overline{X}$ explicitly, but the expected value and variance are the most important, and these we can find by applying Theorem 5.2.6:

$$E(\overline{X}) = \lambda \quad \text{and}\, Var(\overline{X}) = \frac{\lambda}{n},$$

since $E(X) = Var(X) = \lambda$, when X is Poisson(λ).

If $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ is the sample variance, then by 5.2.6,

$$E(S^2) = \lambda,$$

so we see that S^2 is also an unbiased estimator for λ.

See page 335 for a continuation of this discussion.