2.5 Exercises

2.1 In each of the following find the pdf of Y. Show that the pdf integrates to 1.

(a) $Y = X^3$ and $f_X(x) = 42x^5(1-x)$, $0 < x < 1$

(b) $Y = 4X + 3$ and $f_X(x) = 7e^{-7x}$, $0 < x < \infty$

(c) $Y = X^2$ and $f_X(x) = 30x^2(1-x)^2$, $0 < x < 1$

(See Example A.0.2 in Appendix A.)

2.2 In each of the following find the pdf of Y.

(a) $Y = X^2$ and $f_X(x) = 1$, $0 < x < 1$

(b) $Y = -\log X$ and X has pdf

$$f_X(x) = \frac{(n+m+1)!}{n!\,m!}\,x^n(1-x)^m, \quad 0 < x < 1, \quad m, n \text{ positive integers}$$

(c) $Y = e^X$ and X has pdf

$$f_X(x) = \frac{1}{\sigma^2}\,xe^{-(x/\sigma)^2/2}, \quad 0 < x < \infty, \quad \sigma^2 \text{ a positive constant}$$

2.3 Suppose X has the geometric pmf $f_X(x) = \frac{1}{3}\left(\frac{2}{3}\right)^x$, $x = 0, 1, 2, \ldots$. Determine the probability distribution of $Y = X/(X+1)$. Note that here both X and Y are discrete random variables. To specify the probability distribution of Y, specify its pmf.

2.4 Let λ be a fixed positive constant, and define the function $f(x)$ by $f(x) = \frac{1}{2}\lambda e^{-\lambda x}$ if $x \geq 0$ and $f(x) = \frac{1}{2}\lambda e^{\lambda x}$ if $x < 0$.

(a) Verify that $f(x)$ is a pdf.

(b) If X is a random variable with pdf given by $f(x)$, find $P(X < t)$ for all t. Evaluate all integrals.

(c) Find $P(|X| < t)$ for all t. Evaluate all integrals.

2.5 Use Theorem 2.1.8 to find the pdf of Y in Example 2.1.2. Show that the same answer is obtained by differentiating the cdf given in (2.1.6).

2.6 In each of the following find the pdf of Y and show that the pdf integrates to 1.

(a) $f_X(x) = \frac{1}{2}e^{-|x|}$, $-\infty < x < \infty$; $Y = |X|^3$

(b) $f_X(x) = \frac{3}{8}(x+1)^2$, $-1 < x < 1$; $Y = 1 - X^2$

(c) $f_X(x) = \frac{3}{8}(x+1)^2, -1 < x < 1$; $Y = 1 - X^2$ if $X \leq 0$ and $Y = 1 - X$ if $X > 0$

2.7 Let X have pdf $f_X(x) = \frac{2}{9}(x+1)$, $\quad -1 \leq x \leq 2$.

(a) Find the pdf of $Y = X^2$. Note that Theorem 2.1.8 is not directly applicable in this problem.

(b) Show that Theorem 2.1.8 remains valid if the sets $A_0, A_1, \ldots, A_k$ *contain* $\mathcal{X}$, and apply the extension to solve part (a) using $A_0 = \emptyset$, $A_1 = (-1, 1)$, and $A_2 = (1, 2)$.

2.8 In each of the following show that the given function is a cdf and find $F_X^{-1}(y)$.

(a) $F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-x} & \text{if } x \geq 0 \end{cases}$

(b) $F_X(x) = \begin{cases} e^x/2 & \text{if } x < 0 \\ 1/2 & \text{if } 0 \le x < 1 \\ 1 - (e^{1-x}/2) & \text{if } 1 \le x \end{cases}$

(c) $F_X(x) = \begin{cases} e^x/4 & \text{if } x < 0 \\ 1 - (e^{-x}/4) & \text{if } x \ge 0 \end{cases}$

Note that, in part (c), $F_X(x)$ is discontinuous but (2.1.13) is still the appropriate definition of $F_X^{-1}(y)$.

2.9 If the random variable X has pdf

$$f(x) = \begin{cases} \frac{x-1}{2} & 1 < x < 3 \\ 0 & \text{otherwise,} \end{cases}$$

find a monotone function $u(x)$ such that the random variable $Y = u(X)$ has a uniform(0, 1) distribution.

2.10 In Theorem 2.1.10 the probability integral transform was proved, relating the uniform cdf to any continuous cdf. In this exercise we investigate the relationship between discrete random variables and uniform random variables. Let X be a discrete random variable with cdf $F_X(x)$ and define the random variable Y as $Y = F_X(X)$.

(a) Prove that Y is stochastically greater than a uniform(0, 1); that is, if $U \sim$ uniform (0, 1), then

$$P(Y > y) \ge P(U > y) = 1 - y, \quad \text{for all } y,\ 0 < y < 1,$$
$$P(Y > y) > P(U > y) = 1 - y, \quad \text{for some } y,\ 0 < y < 1.$$

(Recall that *stochastically greater* was defined in Exercise 1.49.)

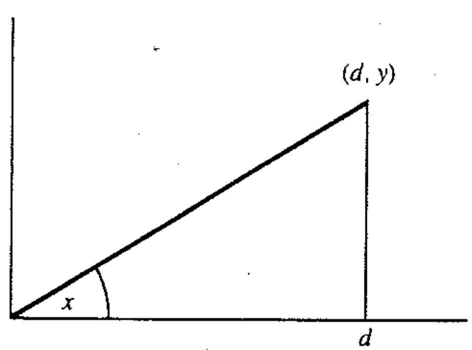
(b) Equivalently, show that the cdf of Y satisfies $F_Y(y) \le y$ for all $0 < y < 1$ and $F_Y(y) < y$ for some $0 < y < 1$. (*Hint:* Let x_0 be a jump point of F_X, and define $y_0 = F_X(x_0)$. Show that $P(Y \le y_0) = y_0$. Now establish the inequality by considering $y = y_0 + \epsilon$. Pictures of the cdfs will help.)

2.11 Let X have the standard normal pdf, $f_X(x) = (1/\sqrt{2\pi})e^{-x^2/2}$.

(a) Find $\mathrm{E}X^2$ directly, and then by using the pdf of $Y = X^2$ from Example 2.1.7 and calculating $\mathrm{E}Y$.

(b) Find the pdf of $Y = |X|$, and find its mean and variance.

2.12 A random right triangle can be constructed in the following manner. Let X be a random angle whose distribution is uniform on $(0, \pi/2)$. For each X, construct a triangle as pictured below. Here, Y = height of the random triangle. For a fixed constant d, find the distribution of Y and $\mathrm{E}Y$.



2.13 Consider a sequence of independent coin flips, each of which has probability p of being heads. Define a random variable X as the length of the run (of either heads or tails) started by the first trial. (For example, $X = 3$ if either TTTH or HHHT is observed.) Find the distribution of X, and find $\mathrm{E}\,X$.

2.14 (a) Let X be a continuous, nonnegative random variable $[f(x) = 0$ for $x < 0]$. Show that

$$\mathrm{E}\,X = \int_0^\infty [1 - F_X(x)]\,dx,$$

where $F_X(x)$ is the cdf of X.

(b) Let X be a discrete random variable whose range is the nonnegative integers. Show that

$$\mathrm{E}\,X = \sum_{k=0}^{\infty} (1 - F_X(k)),$$

where $F_X(k) = P(X \le k)$. Compare this with part (a).

2.15 Betteley (1977) provides an interesting addition law for expectations. Let X and Y be any two random variables and define

$$X \wedge Y = \min(X, Y) \quad \text{and} \quad X \vee Y = \max(X, Y).$$

Analogous to the probability law $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, show that

$$\mathrm{E}(X \vee Y) = \mathrm{E}\,X + \mathrm{E}\,Y - \mathrm{E}(X \wedge Y).$$

(*Hint*: Establish that $X + Y = (X \vee Y) + (X \wedge Y)$.)

2.16 Use the result of Exercise 2.14 to find the mean duration of certain telephone calls, where we assume that the duration, T, of a particular call can be described probabilistically by $P(T > t) = ae^{-\lambda t} + (1-a)e^{-\mu t}$, where a, λ, and μ are constants, $0 < a < 1$, $\lambda > 0$, $\mu > 0$.

2.17 A *median* of a distribution is a value m such that $P(X \le m) \ge \frac{1}{2}$ and $P(X \ge m) \ge \frac{1}{2}$. (If X is continuous, m satisfies $\int_{-\infty}^{m} f(x)\,dx = \int_m^\infty f(x)\,dx = \frac{1}{2}$.) Find the median of the following distributions.

(a) $f(x) = 3x^2, \quad 0 < x < 1$ (b) $f(x) = \dfrac{1}{\pi(1 + x^2)}, \quad -\infty < x < \infty$

2.18 Show that if X is a continuous random variable, then

$$\min_a \mathrm{E}\,|X - a| = \mathrm{E}\,|X - m|,$$

where m is the median of X (see Exercise 2.17).

2.19 Prove that

$$\frac{d}{da}\,\mathrm{E}(X - a)^2 = 0 \Leftrightarrow \mathrm{E}\,X = a$$

by differentiating the integral. Verify, using calculus, that $a = \mathrm{E}\,X$ is indeed a minimum. List the assumptions about F_X and f_X that are needed.

2.20 A couple decides to continue to have children until a daughter is born. What is the expected number of children of this couple? (*Hint:* See Example 1.5.4.)

2.21 Prove the "two-way" rule for expectations, equation (2.2.5), which says $\mathrm{E}\,g(X) = \mathrm{E}\,Y$, where $Y = g(X)$. Assume that $g(x)$ is a monotone function.

2.22 Let X have the pdf

$$f(x) = \frac{4}{\beta^3\sqrt{\pi}} x^2 e^{-x^2/\beta^2}, \quad 0 < x < \infty, \quad \beta > 0.$$

(a) Verify that $f(x)$ is a pdf. (b) Find $\mathrm{E}\,X$ and $\mathrm{Var}\,X$.

2.23 Let X have the pdf

$$f(x) = \frac{1}{2}(1+x), \quad -1 < x < 1.$$

(a) Find the pdf of $Y = X^2$. (b) Find $\mathrm{E}\,Y$ and $\mathrm{Var}\,Y$.

2.24 Compute $\mathrm{E}\,X$ and $\mathrm{Var}\,X$ for each of the following probability distributions.

(a) $f_X(x) = ax^{a-1}$, $0 < x < 1$, $a > 0$
(b) $f_X(x) = \frac{1}{n}$, $x = 1, 2, \ldots, n$, $n > 0$ an integer
(c) $f_X(x) = \frac{3}{2}(x-1)^2$, $0 < x < 2$

2.25 Suppose the pdf $f_X(x)$ of a random variable X is an *even function*. ($f_X(x)$ is an *even function* if $f_X(x) = f_X(-x)$ for every x.) Show that

(a) X and $-X$ are identically distributed.
(b) $M_X(t)$ is symmetric about 0.

2.26 Let $f(x)$ be a pdf and let a be a number such that, for all $\epsilon > 0$, $f(a+\epsilon) = f(a-\epsilon)$. Such a pdf is said to be *symmetric* about the point a.

(a) Give three examples of symmetric pdfs.
(b) Show that if $X \sim f(x)$, symmetric, then the median of X (see Exercise 2.17) is the number a.
(c) Show that if $X \sim f(x)$, symmetric, and $\mathrm{E}\,X$ exists, then $\mathrm{E}\,X = a$.
(d) Show that $f(x) = e^{-x}$, $x \geq 0$, is not a symmetric pdf.
(e) Show that for the pdf in part (d), the median is less than the mean.

2.27 Let $f(x)$ be a pdf, and let a be a number such that if $a \geq x \geq y$, then $f(a) \geq f(x) \geq f(y)$, and if $a \leq x \leq y$, then $f(a) \geq f(x) \geq f(y)$. Such a pdf is called *unimodal* with a *mode* equal to a.

(a) Give an example of a unimodal pdf for which the mode is unique.
(b) Give an example of a unimodal pdf for which the mode is not unique.
(c) Show that if $f(x)$ is both symmetric (see Exercise 2.26) and unimodal, then the point of symmetry is a mode.
(d) Consider the pdf $f(x) = e^{-x}$, $x \geq 0$. Show that this pdf is unimodal. What is its mode?

2.28 Let μ_n denote the nth central moment of a random variable X. Two quantities of interest, in addition to the mean and variance, are

$$\alpha_3 = \frac{\mu_3}{(\mu_2)^{3/2}} \quad \text{and} \quad \alpha_4 = \frac{\mu_4}{\mu_2^2}.$$

The value α_3 is called the *skewness* and α_4 is called the *kurtosis*. The skewness measures the lack of symmetry in the pdf (see Exercise 2.26). The kurtosis, although harder to interpret, measures the peakedness or flatness of the pdf.