Mathematical Foundations for the Common Core
A Course for Middle and Secondary Teachers

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Topic: Why is the graph of a linear equation a straight line?

The Common Core State Standards state that students in eighth grade ought to:

> *Use similar triangles to explain why the slope m is the same between any two distinct points on a non-vertical line in the coordinate plane; derive the equation $y = mx$ for a line through the origin and the equation $y = mx + b$ for a line intercepting the vertical axis at b.(8.EE.6)*

The understanding of lines and linear equations is only one piece of a broader understanding of geometry that is supposed to develop in eighth grade. Also in 8^{th} grade, students are expected to

> *describe and analyze two-dimensional figures and to solve problems. Students show that the sum of the angles in a triangle is the angle formed by a straight line, and that various configurations of lines give rise to similar triangles because of the angles created when a transversal cuts parallel lines. Students understand the statement of the Pythagorean Theorem and its converse, and can explain why the Pythagorean Theorem holds, for example, by decomposing a square in two different ways. (CCSS, page 52.)*

Harel and Wilson (Notices of the AMS, June/July 2011, 823-826) reviewed popular United States high school math textbooks from four different publishers. Concerning linear functions and their graphs, the authors size up the programs as follows:

> *No program produced the basics here. Slope, although defined, is never shown to be well defined. It is never shown that the graph of an algebraic linear function really is a line in the coordinate plane, and it is never shown that a line in the coordinate plane really is the graph of an algebraic linear function. The worst aspect of this was that it seemed the textbook authors were unaware that something was missing.*

Books like these will be of no help in meeting the standard quoted above. U.S. textbooks at the 8^{th} grade level do not appear to be any better.

The components of the geometric understanding that the Common Core standards refer to—parallel lines, transversals, similar triangles, the Pythagorean Theorem and the relationship between lines and linear equations—all fit together in a remarkable way.

What Fermat Said

The following passage is from Fermat's essay "Introduction to Plane and Solid Loci," translated by J. Seidlin, in D.E.Smith, *A Source Book in Mathematics*, McGraw Hill, 1929. This elegant statement (composed around 1630) is one of the earliest modern written discussions of the relationship between linear equations and lines. Fermat uses vowels a, e as variable symbols. Consonants b, c, d stand for fixed numbers.

> *Let NZM be a straight line of given position with point N fixed. Let NZ be the unknown quantity a and ZI (the line drawn to form the angle NZI) the other unknown quantity e.*

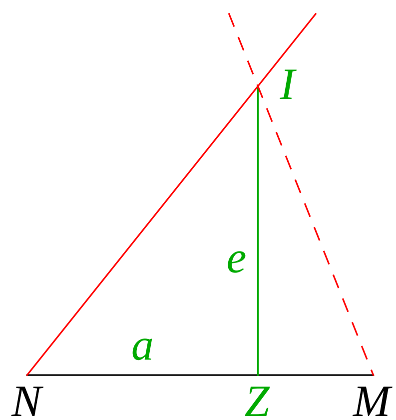

Figure 1.

> *If $da = be$, the point I will describe a line of fixed position. Indeed, we would have*
>
> $$\frac{b}{d} = \frac{a}{e}.$$
>
> *Consequently, the ratio $a:e$ is given, as is also the angle at Z. Therefore, both the triangle NIZ and the angle INZ are determined. But the point N and the position of the line NZ are given, and so the position of NI is determined.*

Fermat's argument is essentially as follows. Suppose that N and M are fixed while Z moves along the ray $\overrightarrow{NM}$ and I moves in the plane in such a manner that the angle $\angle NZI$ and ratio $a:e$ are maintained. Then, the various triangles $\triangle NZI$ that are formed as Z and I move are all similar to one another by the SAS criterion for triangle similarity. Therefore the angle $\angle INZ$ remains fixed. This means that I moves along a fixed ray. In other words, if a and e are given the meaning in the diagram, then the equation $da = be$ describes a ray.

In the next paragraph of his essay, Fermat examines the more general equation

$$c - da = be. \tag{1}$$

Choose a number r so that $c = d\,r$. Then (1) is equivalent to

$$\frac{b}{d} = \frac{r-a}{e}. \tag{2}$$

If the length of NM is r, then MZ is $r - a$. The same argument as before shows that if Z and I move to preserve the angle at Z as well as the equation $\frac{b}{d} = \frac{r-a}{e}$, then I is constrained to lie on the dashed line.

Another approach

The argument presented here does not depend on similar triangles, but on the following fact:

Fact. *Let $Q = (a, b)$ and $Z = (c, d)$ be two points in the plane, both different from the origin $O = (0, 0)$. Then $\angle QOZ$ is right if and only if $0 = ac + bd$.*

Proof. By the Pythagorean Theorem and its converse,

$$\begin{aligned}
\angle QOZ \text{ is right} &\Leftrightarrow |OQ|^2 + |OZ|^2 = |QZ|^2 \\
&\Leftrightarrow a^2 + b^2 + c^2 + d^2 = (c - a)^2 + (d - b)^2 \\
&\Leftrightarrow a^2 + b^2 + c^2 + d^2 = c^2 - 2ac + a^2 + (d^2 - 2bd) + b^2 \\
&\Leftrightarrow 0 = -2ac - 2bd \\
&\Leftrightarrow 0 = ac + bd. \qquad\qquad /////
\end{aligned}$$

Now let ℓ be any line and P_0 be any point on it. Choose t-u coordinates centered at P_0. (Then the x-y and t-u coordinates are related by the equations: $t(P) = x(P) - x(P_0)$ and $u(P) = y(P) - y(P_0)$.)

Choose a single point Q so that QP_0 is perpendicular to ℓ. Let $a := t(Q)$ and $b := u(Q)$. Then

$$\begin{aligned}
Z \in \ell &\Leftrightarrow Z = P_0 \text{ or } \angle QP_0Z \text{ is right} \\
&\Leftrightarrow 0 = a\,t(Z) + b\,u(Z) \\
&\Leftrightarrow 0 = a\,(x(Z) - x(P_0)) + b(y(Z) - y(P_0)) \\
&\Leftrightarrow a\,x(P_0) + b\,y(P_0) = a\,x(Z) + b\,y(Z).
\end{aligned}$$

Thus, $Z \in \ell$ if and only if the x-y-coordinates of Z satisfy

$$C = a\,x + b\,y,$$

where $C = a\,x_0 + b\,y_0$, with $P_0 = (x_0, y_0)$ on ℓ, and $(a + x_0, b + y_0)$ the coordinates of a point Q such that QP_0 is perpendicular to ℓ.