**Order of Operations**

Some arithmetic expressions appear to allow different interpretations. For example, most people will simplify $6-3+2$ as follows:

$$6-3+2 \quad = \quad (6-3)+2 \quad = \quad 3+2 \quad = \quad 5,$$

but a few may read $6-3+2$ as $6-(3+2)$, which equals 1. Most people will simplify $6+3\times 2$ this way:

$$6+3\times 2 \quad = \quad 6+(3\times 2) \quad = \quad 6+6 \quad = \quad 12,$$

but a few will read $6+3\times 2$ as $(6+3)\times 2$, which is 18.

The *order of operations* is a set of rules, often taught is school, that determine how ambiguous expressions are to be interpreted.

**The rules for the order of operations**

I. A string of additions and/or subtractions is to be evaluated from left to right. A minus-sign operates only on the number that is immediately to its right, causing that number to be subtracted from the number that results from evaluating the string that precedes it. Thus,

$$6-1+5-2-3$$

is to be interpreted in the following manner:

$$\boxed{\boxed{\boxed{6-1}+5}-2}-3.$$

The steps in simplifying the expression are:

$$\begin{aligned} 6-1+5-2-3 \quad &= \quad \boxed{\boxed{\boxed{6-1}+5}-2}-3 \\ &= \quad \boxed{\boxed{5+5}-2}-3 \\ &= \quad \boxed{10-2}-3 \\ &= \quad 8-3 \\ &= \quad 5. \end{aligned}$$

II. A string of multiplications and/or divisions is also to be evaluated from left to right. A division-sign operates only on the number that is immediately to its right. Thus,

$$8\div 4\times 5\div 2\times 3$$

is to be interpreted as:

$$\boxed{\boxed{\boxed{8\div 4}\times 5}\div 2}\times 3.$$

III. In a string involving additions and/or subtractions *as well as* multiplications and/or divisions, the multiplications and/or divisions are to be done before the additions and/or subtractions. Thus,

$$7+2\times 6\div 3\times 5-9+5\times 4$$

is to be interpreted as:

$$7 + \boxed{2 \times 6 \div 3 \times 5} - 9 + \boxed{5 \times 4} = 7 + 20 - 9 + 20 = 38.$$

IV. Parentheses may be used to override any of the rules above. Expressions inside parentheses are evaluated first, no matter what operations they involve.

$(1+2)\times 3 = 3\times 3 = 9\,;$ $\quad 5-(3-2) = 5-1 = 4\,;$ $\quad 5-(5-(3-2)) = 5-(5-1) = 5-4 = 1\,.$

When division is represented using a horizontal bar (as in $\frac{3+1}{2}$), the bar effectively creates parentheses around the numerator and denominator:

$$\frac{4+5}{1+2} = (4+5) \div (1+3) = 9/3 = 3.$$

V. Exponents act on the numbers immediately preceding them. Thus, $3 \times 5^2$ is interpreted as $3 \times \boxed{5^2}$. When a minus sign is involved, as in $-5^2$, it is not interpreted as part of the base; $-5^2$ means $-(5^2)$.

**Simplifying arithmetic expressions**

An arithmetic expression is evaluated by attending to the parentheses first. Within them, one performs the multiplications and divisions and then additions and subtractions. If there are several layers of parentheses, the process is iterated, beginning with the innermost parentheses.

When an operation is performed inside a more complex arithmetic expression, a number is written in place of the operation symbol and the numbers on which it acted. This reduces the number of operation symbols. The following example illustrates this. At each stage, we underlined the operation to be performed and the numbers on which the operator acts. There are 5 operation symbols in the first expression, 4 in the next, then 3, 2, 1, and finally none.

$$\begin{aligned}\big(1 + 3 \cdot (3 + \underline{2 \cdot 5})\big)/2 &= \big(1 + 3 \cdot (\underline{3 + 10})\big)/2 \\ &= (1 + \underline{3 \cdot 13})/2 \\ &= (\underline{1 + 39})/2 \\ &= \underline{40/2} \\ &= 20.\end{aligned}$$

When all the operations have been performed, what remains at the end of the process is a number.

**Problems**

1) Read: Hung-Hsi Wu, *"Order of operations" and other oddities in school mathematics*, available at `http://math.berkeley.edu/~wu/order5.pdf`.
2) According to Wu, where do mathematicians use order of operations?
3) According to Wu, the "left-to-right" rule is unnecessary. What is the reason for this?
4) To what uses of order of operations in school math and math assessment does Wu object?
5) What do the Common Core State Standards for Mathematics say about order of operations?