**Coordinates on the line** June 19, 2012

We have seen that to *solve* an equation—or an inequality, or a combination of equations and/or inequalities joined by ANDs and ORs—means to find the numbers which, when inserted in place of the variables, yield a true sentence. The examples involving inequalities that we looked at suggested that it can be useful to depict the solution set geometrically.

## Coordinate systems (plural!) on the line

In several previous lectures, I talked about the number line. To refresh your memory, we started with a line, chose a unit of distance, a point on the line (called the origin) and a direction on the line (called the positive direction). We then labelled the points on the line with numbers in such a way that:

1) the origin was assigned 0,
2) the points on the positive side of the origin were labelled with positive numbers and
3)

$$\left.\begin{array}{r}\text{the distance between any two}\\ \text{points, as measured by the unit}\end{array}\right\} = \left\{\begin{array}{l}\text{the absolute value of the}\\ \text{difference between their labels.}\end{array}\right.$$

Once points have been labelled in this way, we have a *coordinate system* on the line. Each point has a name that is a number, and each number names a point. Different numbers name different points and different points have different number-names.

To make the coordinate system, we had to make some choices—an origin, a positive direction and a unit of length. But wait! Though we needed to make these choices in order to have a coordinate system, we we had some leeway. We were not required to make them in a particular way, but only to settle on some choices that we made explicit. What if, for another purpose, someone else decided to use a different origin, a different direction and/or a different unit of length? How can we relate her coordinate system to ours? This is not an unusual problem. You already saw an instance of it in the Fahrenheit and Celsius markings on the thermometer.

*Example.* In the Pink Cadillac problem, some people noticed that the solution can be obtained easily if we choose a spatial coordinate system that has the Cadillac as origin and a temporal coordinate system that has the moment that the Toyota leaves as the the origin. The spatial coordinate of the Toyota is its signed distance from the Cadillac. This starts out negative, but increases at a rate equal to the difference in speeds and eventually hits 0 when the two cars meet.

To be illustrate, let us recall the latest version of the problem: A pink Cadillac leaves Oklahoma City at 6AM headed west on I-40 with the cruise control set at $V$ mph. A federal agent in a green Toyota follows, leaving $d$ minutes later and traveling $W$ mph. Does the Toyota catch up? If so, when and where?

*Solution.* Let $t$ be the variable that represents the number of hours elapsed since the Toyota left. Let $s$ be the variable that denotes signed distance in miles from the Cadillac, with positions to the west of the Cadillac having positive coordinates. Let $s_T(t)$ be the function that gives the Toyota's position at time $t$. Now, the Toyota is $V\frac{d}{60}$ miles east of the Cadillac at time $t = 0$, which implies that $s_T(0) = -V \cdot \frac{d}{60}$. Moreover, $s_T$ is linear, with slope equal to the rate at which the distance between the two cars is changing, namely $W - V$. Thus,

$$s_T(t) = (W - V)t - V\frac{d}{60}.$$

The cars meet at whatever times $t$ solve the equation $0 = s_T(t)$. But

$$0 = s_T(t) \;\Leftrightarrow\; 0 = (W - V)t - V\frac{d}{60} \;\Leftrightarrow\; t = V\frac{d}{60(W - V)}$$

Conclusion: The cars meet $V\frac{d}{60(W-V)}$ hours after the Toyota departs.

*Example. The following problem shows that choosing the right coordinate system can turn a problem that appears hard into a very simple one.* A man is rowing a boat upstream in a river with constant effort. As he passes under a bridge his hat falls off. He continues rowing for 10 minutes before noticing. Immediately, he turns around and rows with the same effort until he reaches his hat one mile downstream from the bridge. How fast is the river flowing?

**Relating different coordinate systems**

It is very useful to create a symbol to denote the number associated with a point. A common practice is to use the symbol $x$ for that purpose. If $P$ is a point on the line, then $x(P)$ denotes the number that our coordinate system assigns to it. The expression $x(P)$ might seem peculiar to you. In high-school mathematics, people often talk about numbers and positions as if there were no distinction between them; $x$ is a point *or* a number, and how you view it just depends on what you're doing. If high school curricula can get along this way, why do I insist on bothering you with names like $P$ for points, and why should you tolerate an unnecessarily elaborate expression like $x(P)$? I plan to show you that this is a small price for a great convenience.

Suppose a second coordinate system on the same line is chosen. We need a different name for the number assigned to a point, so let us use the symbol $w$ for this purpose. What is the relationship between $w(P)$ and $x(P)$? In many cases, the answer is quite obvious:

a) If the *only* difference between the systems is the choice of direction, then

$$w(P) = -x(P)$$

for all points $P$.

b) If the two systems differ *only* in the location of the origin, then $w(P)-w(Q) = x(P)-x(Q)$ for all points $P$ and $Q$. In particular, if $O_x$ is the origin for the $x$-system, then $w(P)-w(O_x) = x(P)-x(O_x) = x(P)$. Thus,

$$w(P) = x(P) + b,$$

where $b = w(O_x)$

c) Finally, if a different unit is used for the $w$ system, but the origin and direction are the same, then

$$w(P) = m\,x(P),$$

where $m$ is *conversion factor*, which tells us the length of the $x$-unit when measured by the $w$-unit, i.e., the number of $w$-units per $x$-unit.

**Problem.** Find a general formula for converting from one coordinate system to another; it should look like this:

$$w(P) = m\,x(P) + b.$$

What is the meaning of $m$ and $b$? (Hint: To find $m$, pick two points $P$ and $Q$ and think about the meaning of the differences $x(P) - x(Q)$ and $w(P) - w(Q)$.) Will $b$ still be the value of $w(O_x)$ (as it was in part b), above) when the unit of length changes? Why or why not?

**Expressing functions in different coordinate systems**

**Problem.** A stone is thrown upwards from ground level. When it reachers a height of 160 feet, it is still traveling upwards at 24 feet per second. When was it thrown? When does it return to the ground?

*Background.* The laws of motion (which the physicists have supplied for us) tell us that the height $t$ second after reaching 160 feet is:

$$h = -16t^2 + 24\,t + 160.$$

The equation is a valid description of the height from the moment the stone was thrown until the moment it returns to the ground. Here, $t$ is a coordinate system on the time line. We used seconds as the unit, placed 0 at the time the object was at 160 feet and let the direction be the natural one. The object was thrown at a time with a negative coordinate.

*Solution.* We can simplify the equation by choosing a unit of time that has duration 1/4 second. Keep the same origin and direction, let $w$ be the coordinate system with these units. For any point $Z$ in time,

$$w(Z) = 4t(Z).$$

(Why is it NOT $w(Z) = \frac{1}{4}t(Z)$?) Then

$$h = -(4t)^2 + 6(4t) + 160 = -w^2 + 6\,w + 160.$$

We can simplify the equation further by choosing a coordinate system that will cause the linear term in the expression for the function to vanish. To do this, we create yet another coordinate system $u$ with its origin placed strategically. Let

$$u(P) = w(P) - 3.$$

In other words, the $u$-clock, like the $w$-clock, is marked in quarter seconds, but it starts when the $w$-clock reads 3. (The reason 3 was chosen will be apparent in a moment.) Then

$$\begin{aligned} h &= -w^2 + 6\,w + 160 \\ &= -(u+3)^2 + 6\,(u+3) + 160 \\ &= -u^2 - 6\,u - 9 + 6\,u + 18 + 160 \\ &= -u^2 + 169 \\ &= (13-u)(13+u). \end{aligned}$$

In the $u$-coordinate system, we see that $h = 0$ when $u = \pm 13$. The object was thrown when $u = -13$; it returns to the ground when $u = 13$. Since the $u$-coordinate system has its origin 3/4 of a second *after* the $w$-origin, the object was thrown 10/4 seconds *before* the moment it reached 160 feet (i.e., when $t = 2.5$), and it will hit the ground 16/4 second after that moment (i.e., when $t = 4$).

**Project**

In the problem that we solved in the last section, we made two coordinate transforms to convert a quadratic equation of the form $a\,x^2 + b\,x + c = 0$ into one of the form $u^2 - s = 0$. The latter form is easy to solve, and the equations for the coordinate transforms enable us to take a solution for the latter back home to solve the former. In the work abve, we had specific numerical constants in place of the "generic constants" $a$, $b$ and $c$. The same process that we used above can also be carried out with the generic constants. The task posed here, then, is fourfold. First, carry this out. Second, write a careful exposition that would be understandable and informative to a high school class,. Third, based on what you understand of the process and on the goals of the Common Core Algebra standards, decide what learning goals relative to the highschool curriculum are involved here and state them. Look one or two big, important ideas, not a long, mindless list of trivialities. Finally, create an assessment problem that would determine if the big ideas had been conveyed.