Mathematical Foundations for the Common Core
A Course for Middle and Secondary Teachers

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Conversion Factors, and some speculations about student errors June 6, 2012

What is a conversion factor? Suppose we are doing some work, and we have some materials that are measured in inches and and some that are measured in centimeters. We might know the measure of an object with respect to one unit, but need to know the measure with respect to the other. We can achieve this by multiplying by an appropriate *conversion factor*.

There are 2.54 centimeters in one inch,[1] and therefore, 2.54 is the conversion factor that changes a number of inches into the number of centimeters that spans the same length. If a piece of wire is carefully cut to exactly 69.5 inches, then its length in centimeters is

$$2.54 \times 69.5 = 176.53.$$

We can think of 2.54 as the numerical part of a quantity with label "centimeters per inch," or $\frac{\text{cm.}}{\text{in.}}$, and we have the following equation involving quantities (rather than just numbers):

$$2.54\frac{\text{cm.}}{\text{in.}} \times 69.5\,\text{in.} = 176.53\,\text{cm.}$$

It does not matter how big or small the objects are. If we measure with both units over and over, then as long as the measurements are accurate, we always get 2.54 when we divide the measure in centimeters by the measure in inches.

If an object measures x inches and also measures y centimeters, then the *numerical parts* of the quantities (the numerical parts are the x and the y) satisfy the relation

$$2.54\,x = y.$$

We can also write an equation that makes the quantities explicit:

$$2.54\frac{\text{cm.}}{\text{in.}} \times x\,\text{in.} = y\,\text{cm.}$$

[1] The *international inch* is by definition *exactly* 2.54 centimeters. There are other units that go by the name "inch" and are very close to the international inch; see:
`http://www.nist.gov/pml/wmd/metric/upload/SP1038.pdf`

Test your knowledge! The following problems illustrate some of the difficulties that one may run into when using conversion factors—or when working with rates generally.

1) *Sally is bicycling Ireland, where distances are measure in kilometers. She traveled 80 kilometers today. She wants to know how many miles that is. She knows that 6 miles is a bit less than 10 kilometers; what can she conclude based on this knowledge?*

2) *On June 18, 2011, the currency exchange rate was 1 euro = 1.4315 US dollars. If Sally had D dollars on that day, then what equation should she use to find out the number E of euros that her money was worth?*

   a) $D = 1.4315\,E$

   b) $E = 1.4315\,D$

3) *Today (June 6, 2012), 1 euro = 1.2445 US dollars. All of Sally's money is in US dollars. If she plans to spend her money in Ireland (where the euro is used), is she better off now than she was a year ago?*

Possible sources of trouble in dealing with quantities

(1) In the Plato's Trail Mix problem, if we go directly from words to symbols we will be misled, because the *order of the words* in the statement, "One gets ten raisins for every three nuts," is the same as the *order of the symbols* in the *incorrect* equation, "$10R = 3N$."

(2) A student who is learning the language of algebra but who has not yet mastered it might interpret "$10R = 3N$" as the statement, "Ten raisins gives three nuts," or "Getting ten raisins in a serving means getting three nuts." This parallels the mistake in 1), but it does not originate from direct pattern-matching. Instead, this error is the result of a reasonable, but incorrect, interpretation of the symbols, an interpretation under which R and N are taken as abbreviated *names for the objects* rather than as *names for numbers that measure collections of objects*. This interpretation is aided and abetted by an interpretation of the equals sign that is common among students. Experience in arithmetic encourages students to view it as shorthand for "and so you get", or "produces" or "leads to", like the horizontal bar at the base of a column addition problem. A student who interprets the symbols as names for things rather than numbers could take the equals sign as a signal that two collections of objects are closely related, in the sense that they occur together. A student who takes the letters as names for things could not interpret the equals sign in the standard algebraic way (as part of a statement relating two numbers).

3) Some language in common use tends to lead to confusion. Many people say things such as, "To convert inches to centimeters, multiply inches by 2.54." What is meant by "inches" in this statement is the *number of inches in the thing measured*. (A learner who does not recognize this might misunderstand the statement to mean that some collection of inches should increased by a factor of 2.54.) Similarly, some people will use "raisins" to mean "the number of raisins present"—that is, they use "raisins" to mean exactly what R is intended to mean. This can lead to difficulties in translating form

the problem statement to a mathematical statement, for with the words "10 raisins" appearing in the problem, it is easy to pass to the expression $10R$. Again, there is a close relationship to the original, word-order-related problem in (1), but as in (2) there is a rational reason for the misinterpretation. Here, however, no misconstruction of meaning of R is involved. The problem comes from the two different but legitimate meanings of "raisins," which can either be part of a name for a collection of things, or may refer to the number that measures the collection.

4) The potential for standard language to mislead is also present in the way many people speak about ratios. In responding to the Plato's Trail Mix Problem), some people will say:

   The ratio of raisins to nuts is 10 to 3.

   This is might be presented in the following form:

$$\frac{\text{raisins}}{\text{nuts}} = \frac{10}{3}.$$

   In this statement, the word "raisins" must be understood to refer to the number of raisins present in a portion of trail mix, and "nuts" to refer to the number of nuts present. In other words, here "raisins" and "nuts" are playing the roles of the variables R and N. So far, this is not a problem. But if we translate the ratio statement into multiplicative form:

$$3\,\text{raisins} = 10\,\text{nuts}$$

   it appears that we are equating two quantities (3 raisins and 10 nuts), rather than equating two numbers (3 times the number of raisins present and 10 times the number of nuts present). The problem here is made clearer if we consider what one might say about inches and centimeters. If we let "inches" stand for the number of inches that a thing measures and let "centimeters" stand for the number of centimeters that it measures, then we can correctly say:

$$\frac{\text{centimeters}}{\text{inches}} = \frac{254}{100}.$$

   If we rewrite this in multiplicative form, we get:

$$100\,\text{centimeters} = 254\,\text{inches}.$$

   This is **correct only if** we interpret the expressions "inches" and "centimeters" as names for numbers, "centimeters" referring to the *number* of centimeters that an object measures and "inches" referring to the *number* of inches that it measures. But the statement is **incorrect** when we view it as saying that something 100 centimeters long is also 254 inches long, as we would almost surely do if we were to see this statement outside the context of the present discussion.

5) Yet another kind of error can occur in the context of monetary conversions. The variable symbols E and D might be misinterpreted as references to the *worth of a*

single unit, rather than to *a number of units*. We might measure the worth of a thing by the number of grams of silver for which it may be exchanged. If we write e for the value of a euro and d for the value of a dollar (in grams of silver), then $e = 1.4315\,d$. But the E and D do not stand for the worth of the euro and the dollar, they stand for the numbers of euros and dollars that must enter a fair exchange, and these are related by the equation $D = 1.4315\,E$. Like the other errors, this one involves a reasonable but unexpected interpretation of the symbols involved.

The kinds of problems that we have discussed may occur in anyone's thinking. *"Stop, check and be sure!"* is the mathematical habit of mind that saves us, and this habit is one of the most valuable things that we can encourage among those whom we teach.

The problem for teachers is much more profound. There are errors that might be caused by the visual patterns, there are errors that might be related to habits of interpretation, and there are errors that are related to the language maze that surrounds us. The teacher needs to be able to detect the first two kinds of errors when they occur and correct them. The teacher also needs to provide a language environment that is free from the oddities of colloquial language and finally, the teacher must strive to assure that learners are equipped to deal with the confusions that they will certainly encounter when they are not in a classroom where language can be carefully cared for.