Mathematical Foundations for the Common Core
A Course for Middle and Secondary Teachers

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Exposition: Measure numbers **M6302 June 4, 2012**

Each act of measuring involves: 1) the thing measured, 2) the unit by which it is measured and 3) the number that comes out in the end. That last thing is called the "measure number." The act of measuring involves something like division, but it begins with things rather than with numbers. When we measure the length of a piece of rope, we find out how many times the measuring stick fits end-to-end along the rope, and that number is the measure number (*of* the rope *in* measuring-stick units).

In ancient times, this was called a ratio. The measure of the rope is the *ratio* of the rope to the measuring stick. Today, we come across many different opinions about what a ratio "really" is. It does not matter what we call it, but we must bear in mind that measurement is a passage from things to numbers. It is a process where things go in and numbers come out.[2]

In order for anyone to know the meaning of the measure number that comes out of a measurement process, one needs to know what went in. Therefore, the name of the unit is written next to the measure number. It records the unit used. Mathematicians have shown that such "unitized" or "dimensioned" numbers may be treated as special kinds of mathematical objects, similar to pure numbers, but distinct from them. This does not mean, however, that we *must* view them that way. A different approach is to treat measure symbols, such as "inches" or "pounds," as commentary that is supplied in order to link a measure number back to essential information about the measurement process from which it arose. For the purposes of teaching, it is not necessary to make a choice between these philosophical alternatives, but only to assure that learners use correct rules and are able to explain how they are using them, and this is all that I will be attempting in these notes.

A conversion factor is simply the measure number that arises when one unit is measured by another; it is the ratio of one unit to another. Since conversion factors are always

[2] Isaac Newton suggested that this is exactly what numbers are—the things that result when we measure. In his *Universal Arithmetick*, he wrote, "By Number we understand not so much a multitude of Unities as the abstracted Ratio of any Quantity to another Quantity of the same Kind, which we take for Unity." Even earlier, John Wallis was explicit is the idea that in taking a ratio (or making a measurement), we pass from the genus of magnitudes to the genus of number: "When a comparison in terms of ratio is made, the resultant ratio often [namely with the exception of the 'numerical genus' itself] leaves the genus of quantities compared, and passes into the numerical genus, whatever the genus of quantities compared may have been. (John Wallis, *Mathesis Universalis*, translated and quoted in J. Klein, *Greek Mathematical Thought and the Origin of Algebra*. Cambridge. Mass: MIT Press. 1968.)

used in a context where two different units come into play, conversion factors include three pieces of information: a measure number, the unit doing the measuring , and the unit being measured. For example, the conversion factor "2.54 centimeters per inch" contains the information that when an inch is measured using centimeters as a unit, the measure number that results is 2.54—254 centimeters fit in 100 inches with noting left over. When I report the measure of an ordinary thing—e.g., when I say that my pencil measures 18 centimeters—I report the measure number and the unit in the customary manner. When I tell you a conversion factor, I report the measure number and the unit in the same manner, but I also include the phrase "per inch." If I were planning to use my pencil to measure things, then I might want to be able to convert from pencils to centimeters, so I would report a conversion factor of "18 centimeters per pencil."

Multiplying ratios

The ratio of A to B times the ratio of B to C is equal to the ratio of A to C:

$$\frac{A}{B} \cdot \frac{B}{C} = \frac{A}{C}.$$

This is clearly true if A, B and C are numbers. It is also true if A, B and C are things with the same measurable attribute, and the symbol A/B is the measure of the top by the bottom:

$$\text{(pencil measured by inch)} \cdot \text{(inch measured by cm.)} = \text{pencil measured by cm.}$$

If measure numbers with their units are written, then this appears

$$\text{(7.09 inches)} \cdot \text{(2.54 cm. per inch)} = \text{(18.0 cm.)}$$

The conversion factor 2.54 is not the ratio of a centimeter to an inch, even though its "dimension" is cm./inch. It is the ratio of an inch to a centimeter, or the "value of an inch measured in centimeters."

Changing coordinates

Much of this is relevant to a change of coordinates by a change of scale. The x-coordinate of a point on a line is the signed measure of distance of that point from the origin of the x-system, using the unit of the x-system as a standard. If the u-system has the same origin, the equation that relates the x-system and the u-system involves a conversion factor:

$$u(P) = c \cdot x(P).$$

We can think of the conversion factor here as given in "u-units per x-unit," i.e., as a measure of the x-unit by the u-unit. Thus, the u-unit is smaller than the x-unit if and only if the absolute value of the conversion factor is larger than 1.

For example, if we compare the graph of $y = \sin x$ to the graph of $y = \sin 3x$, then the latter can be viewed as the graph of $\sin u$, where the u-system has a smaller unit than the x-system. Since the u-unit is smaller, the graph oscillates more rapidly.