Mathematical Foundations for the Common Core
A Course for Middle and Secondary Teachers

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Measurement, Ratio and Number Systems: A Unifying Theme in K-8

1. The K-5 Measurement Strand

The *Measurement and Data* domain of the *Common Core State Standards for Mathematics* comprises a progression of learning goals that spans Kindergarten to Grade Five. By the end of Grade Five, students are expected to be able to measure length, area, volume, mass, time and angle, convert between different units and interpret the meaning of arithmetic operations applied to *quantities*, the numbers-with-labels that are produced when a thing is measured.[1] After Grade Five, the core ideas in the *Measurement and Data* domain are developed further within the *Ratios and Proportional Relationships* domain in Grades Six and Seven, within the *Number System* domain in these grades and beyond and within the *Number and Quantity* standard in high school.

Measurement, ratio and proportion and number systems are tied together not only in the conceptual architecture of the standards, but also in the history of mathematics. From ancient times all the way up to the early 20th century, Book V of Euclid's *Elements* served as starting point for thinking about these topics and as a guide for the teaching of them. The ideas in this book influenced our understanding of the measurement process, our concepts about ratio and proportion, the logical foundations of the real number system and the manner in which we teach all of these things. Book V helps us to grasp the basic underpinnings of these notions not just in logical way, but as a stratum in the evolution of ideas, and helps us to see that what the Common Core expects students to learn is not just a collection of useful and important ideas, but a tight, coherent structure held together by deep principles and traditions.

In a famous paper of 1901,[2] German mathematician Otto Hölder set out to describe with utmost precision the structure that an attribute must possess in order for it to be measurable and, based upon this precise description of magnitudes, to give an account of the measurement process itself. The description of magnitudes that Hölder arrived at was summarized in his Axioms of Magnitude. It was his goal, he wrote, to deduce from these axioms not only that each magnitude, A, measured by any other magnitude, B, gives a definite real number, but also, that the numerical sum of the two real numbers obtained by measuring magnitudes A and C should be *equal to* the real number obtained by measuring the magnitude resulting from the combination of A and C *as magnitudes*, provided that the three magnitudes, A, C, and their combination, are measured by the same unit. For

[1] e.g., 1609 meters, 60 seconds, 19.32 grams; see [CC], page 58.

[2] O. Hölder, Die Axiome der Quantität und die Lehre vom Mass, *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematisch-Physikaliche Classe* 53, 1-64.

the proofs, he drew directly from the theory of ratio and proportion set out by Euclid in Book V of the *Elements* .

There are many accounts of measurement that are quite different from than that of Hölder. In an attempt to provide a foundation for measurement in the behavioral sciences, psychologist Stanley Smith Stevens proposed that measurement might be made on *nominal*, *ordinal*, *interval* or *ratio* scales. According to Stevens, "measurement, in the broadest sense, is defined as the assignment of numerals to objects or events according to rules."[3] However, measurement in any sense broader than that described by Hölder has diminished mathematical meaning. Even within the behavioral sciences, there are serious questions about the meaning of measurements that do not conform the the Euclidean model.[4]990.

1.1. What did Euclid say?

We will examine what Euclid said (or clearly assumed) about measurement and the entities that are measured. After making these ideas clear, we will compare them with what is expected in the Common Core.

1.1.1. Vocabulary and Basic Assumptions

The Euclidean theory of measurement, as given in Book V and interpreted by Hölder, is built upon the following notions:

1. A *magnitude* is something that has a continuous quantitative attribute. Things that have length, area, volume, mass, duration, and so on are magnitudes.
2. Magnitudes come in *kinds*. Lengths—i.e., things with length—form a kind. Areas—i.e., things with area—form a kind. Masses form a kind, durations form a kind, etc.
3. Two magnitudes of the same kind, we can be *compared*: we can tell if they are equal (with respect to the attribute that determines their kind), and if the are not equal, we can tell which is larger. As a result, given several objects of a given kind, we can *order* them from smallest to largest. Magnitudes of different kinds cannot be compared.[5]

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[3] S. S. Stevens, On the theory of scales of measurement. *Science* 103 (2684), 1946. 677–680.

[4] J. Michell, *An introduction to the logic of psychological measurement*. Hillsdale N.J., Erlbaum. 1

[5] Some misunderstandings that arise in grade-school mathematics are related to the ability to recognize kinds. For example, misconceptions concerning perimeter and area may be arise from having a generalized notion of "size" for plane figures in which the linear dimensions and the area are not differentiated. It is interesting that this kind of error was observed thousands of years ago by Plato. In *Laws* VII, lines 819-820, the Athenian announces that he has witnessed a form of ignorance among his countrymen that seems "more worthy of a stupid beast like the hog than of a human being," and when he is questioned about the nature of this ignorance he reveals that it lies in the belief that lines might be compared to surfaces or surfaces to volumes. It seems that conceiving of perimeter and area in the right way is related to some very fundamental sticking points.

4. We can perform the following operations on magnitudes:
   - *Add.* Given two objects of a kind, we can add them to make a larger thing of the same kind. For example, we can put things with length end-to-end; we can put two masses together in the pan of a balance, we can join two time intervals, etc.
   - *Duplicate and multiply.* We can make copies of a thing—as many as we like—and we may add a given magnitude to itself over and over to form a multiple. If A is a magnitude and we add to it a copy of A, we call the result $2A$. If we add to that another copy of A, we get $3A$. If we add together m copies of A, we call the result mA.
   - *Subtract.* A smaller magnitude may be removed from larger one of the same kind.
5. The operations are "well-behaved":
   - Addition is not sensitive to the order in which the parts are joined or assembled, i.e., it is *associative* $A+(B+C) = (A+B)+C$ and *commutative* $A+B = B+A$.
   - Addition of the same magnitude to two others *preserves the relative size*. In other words, if A is less than B, then $A + C$ is less than $B + C$. The same is true of subtraction; if A is less than B, then $A - C$ is less than $B - C$. Also, if A is the same as B with respect to some quantitative attribute and the same magnitude is added to (or subtracted from) both, then the resulting magnitudes are the same.

Comments:

i) The first item in the list above does not actually define what a magnitude is, but only provides an anchor for thinking, associating the concept with familiar examples. The real meaning of the term is stated in items 3, 4 and 5. To summarize: 3) says that the magnitudes in a kind can be compared with one another, 4) says that the magnitudes in a kind can be duplicated and added and, under some circumstances, subtracted, and finally 5) says that comparison and addition satisfy certain "stability conditions," which assure that different ways of adding yield the same outcome, and that a comparison between two magnitudes will give the same result before and after adding the same magnitude to both.

ii) The manner in which comparisons are performed and the procedures for adding may be complicated. Areas, for example, may be cut up by straight lines and the pieces rearranged without overlap and the new area formed in this fashion is the same as the old. Area A is less than area B if A can be cut up and rearranged into a figure that fits inside of B.

1.1.2. The Measurement Process

The act of measuring is a systematic comparison that requires assigning different roles to two magnitudes of the same kind. One object *gets measured* and the other *does the measuring*. In the following, we shall call often call the former X and the latter U. We say, "X is measured by U." Sometimes we call U the *unit*.

Measuring involves counting the number of units that "fit within" the object being measured, where the meaning of "fit within" depends on the attribute being measured. In the case of length—which is the only kind of magnitude addressed in first grade—one counts how many units may be placed end-to-end along the object being measured without exceeding it. In the case of weight—which is taken up in third grade—one may place the object being measured in one pan of a balance, and then count how many replicas of the unit may be placed in the other pan before the balance tips. The measurement process produces a number that summarizes the relationship between X and U. If X is equal to exactly m Us, that is, if $X = mU$, then we say, "The measure of X by U is m," or "X measured by U is m." If X is greater than mU and less than nU, then we say, "The measure of X by U is between m and n."

Procedurally, the measurement of X by U may be performed in different (but equivalent) ways. One might add copies of U over and over, forming $2U$, $3U$, $4U$, etc., and compare the successive magnitudes to X to determine the largest that does not exceed X. This is in effect what happens when we measure with a ruler, for the ruler is a physical object on which the lengths U, $2U$, $3U$, etc. are represented and labeled. Another way to measure is to subtract copies of U over and over from X, counting as one goes until it is impossible to subtract any more. For example, to find out how many cups of water there are in a container, we might start with a full container and then fill cups from it until we have used up all the water, counting how many cups we've filled as we subtract cups repeatedly.

If X is not equal to a whole number of Us, then the measurement process still leads to a number, but the process by which the number is found is more complex. We will examine this in a subsequent section.

1.2. Connections to the Common Core

We quote directly from the text of the *Common Core Standards* for the Measurement and Data domain, starting in Kindergarten

> **Measurement and Data** **K.MD**
>
> Describe and compare measurable attributes.
>
> 1. Describe measurable attributes of objects, such as length or weight. Describe several measurable attributes of a single object.
> 2. Directly compare two objects with a measurable attribute in common, to see which object has "more of"/"less of" the attribute, and describe the difference. For example, directly compare the heights of two children and describe one child as taller/shorter.
>
> (*A second cluster, that contains an item related to data, not measurement..*)

The point here is to develop the ideas that objects have measurable attributes of various kinds, such as length or weight, that two objects may be compared to determine whether or not they are equal with respect to a measurable attribute that they share, and if they are unequal, to find which is greater. These ideas correspond to the first three points of the Euclidean theory of measurement, as we have described above.

The domain continues in Grade One:

Measurement and Data **1.MD**

Measure lengths indirectly and by iterating length units.

1. Order three objects by length; compare the lengths of two objects indirectly by using a third object.
2. Express the length of an object as a whole number of length units, by laying multiple copies of a shorter object (the length unit) end to end; understand that the length measurement of an object is the number of same-size length units that span it with no gaps or overlaps. Limit to contexts where the object being measured is spanned by a whole number of length units with no gaps or overlaps.

(*There are two additional clusters, but they relate to data, not measurement.*)

The first item is an elaboration of the comparison theme, completing point (3) of the Euclidean theory. The second item introduces the ideas of adding and duplicating units, which appear under point (4) of the Euclidean theory. This gets to the core of the measurement concept. One object may be measured by another with respect to an attribute that they share in common.

In subsequent grades, the principles of measurement are extended to a variety of quantitative attributes, including time, area, volume, mass and angle. Other ideas that are built upon the foundational ideas of Kindergarten and First Grade include the use of fractional units, conversions between units, and the manner in which arithmetic operations applied to measure numbers reflect physical operations that are performed on objects that are measured.

In Grade Two, students are expected to acquire sufficient experience using standard units of length (such as inches, feet, centimeters, meters) to internalize the meaning of these units. Evidence for internalization would be the ability to estimate. Second-graders are also expected to understand and use some basic connections between measurement and arithmetic (i.e., the isomorphism between operations on magnitudes and operations on the numbers), and to begin using the number line as a mental model for arithmetic. In Second Grade we have:

Measurement and Data **2.MD**

Measure and estimate lengths in standard units.

1. Measure the length of an object by selecting and using appropriate tools such as rulers, yardsticks, meter sticks, and measuring tapes.
2. Measure the length of an object twice, using length units of different lengths for the two measurements; describe how the two measurements relate to the size of the unit chosen.
3. Estimate lengths using units of inches, feet, centimeters, and meters.
4. Measure to determine how much longer one object is than another, expressing the length difference in terms of a standard length unit.

Relate addition and subtraction to length.

5. Use addition and subtraction within 100 to solve word problems involving lengths that are given in the same units, e.g., by using drawings (such as drawings of rulers) and equations with a symbol for the unknown number to represent the problem.
6. Represent whole numbers as lengths from 0 on a number line diagram with equally spaced points corresponding to the numbers $0, 1, 2, \ldots$, and represent whole-number sums and differences within 100 on a number line diagram.

(*There are two additional clusters, but they relate to data, not measurement.*)

In Grade Three, students extend the ideas of measurement to time, liquid measure (capacity), mass and—most importantly—to area. Although the measurement of area is based on the ideas that we have discussed, the comparison and addition of areas is more complex. Therefore, we will devote a separate section to this. The more basic measurement topics in third grade are as follows.

Measurement and Data **3.MD**

Solve problems involving measurement and estimation of intervals of time, liquid volumes, and masses of objects.

1. Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram.
2. Measure and estimate liquid volumes and masses of objects using standard units of grams (g), kilograms (kg), and liters (l). Add, subtract, multiply, or divide to solve one-step word problems involving masses or volumes that are given in the same units, e.g., by using drawings (such as a beaker with a measurement scale) to represent the problem.

(*3.MD will continued later.*)

In Grade 4, connections between measurement and arithmetic are developed, with a focus on making inferences about measurable attributes by using arithmetic to convert from one unit to another or to pass from measures of length to measures of area. Angle measure is also introduced, with explicit attention to the analogies to length measure. In Grade 5, the themes from Grade 4 are developed further, brining the full extent of the arithmetic learned up to this point to bear on situations involving measurement. Students do not merely solve problems. They are expected to provide reasoned accounts of manner in which arithmetic operations *represent* physical operations, e.g., relating volume to length and area via multiplication and relating addition of measure numbers to the physical combination of measured objects.

The Measurement and Data domain in Grades 4 and 5 involve significant connections to other domains, and they are far more elaborate and complex than in grades K-3. Therefore, we will delay discussing them in detail until a future section.

1.3. Problems for teachers

1. List all the different kinds of things that students learn to measure by fifth grade and the grades in which they study them. In each case, provide a direct link to the basic measurement process by showing how the measure of an item with respect to a given unit may be understood as a count of the units within the thing measured. (For example, in the case of angles, the standards say, "An angle that turns through 1/360 of a circle is called a "one-degree angle," and can be used to measure angles." How do you count the number of unit angles in a given angle? Or, in the case of volumes, a cube that is one centimeter on a side is used as a unit. How does one go about counting the number of such units in a three-dimensional body?)
2. When you measure the length of a desk with the span of your hand, in addition to the desk itself and your hand, various multiples of your hand length are created. Describe the procedure precisely. What comparisons do you make? How do you know when to stop? How does this lead to a number? (This problem is asking you to take extraordinary care in making the descriptions precise, as in the well-known "explain to a space alien how to make a peanut butter and jelly sandwich" exercise.)
3. "Capacity" refers to the amount of liquid that fits in a container. A tablespoon is a common measure of capacity. Using a tablespoon, how would you measure the capacity of a drinking glass? In what ways is measuring capacity similar to measuring length. Break this down into the precise steps that are followed in both cases, and make a systematic analogy.
4. When a measurement is made, one object *gets measured* and another object *does the measuring*. When we divide numbers, one number *gets divided* and another number *does the dividing*. Is there a significant analogy here, or is the similarity coincidental? Explain.
5. Here is a quotation from the teacher commentary of an *Annenberg Learner* workshop:

    > A measurable property is a property that can be quantified using some kind of unit as a basis. For example, length is measurable, since there is a unit of length (an inch, a centimeter, etc.) and we are counting or measuring the number of units in our object. A non-measurable property is one without a standard unit. When we combine objects with a measurable property, the property must increase.
    >
    > ...
    >
    > If you are having difficulty sorting the attributes, consider which attributes can be quantified. For example, the texture of a rock (e.g., smooth, bumpy, rough) isn't quantifiable using any of the standard units we know; in contrast, the weight of the rock is quantifiable and can be measured in ounces or grams.
    >
    > Another suggestion is to see what happens when you combine your object with another, similar object. If the attribute is measurable, then it will increase when the objects are combined. For example, when you combine two rocks, the texture won't increase or change in any way, but the weight certainly will. (Source: This is Problem A2, on the web

page:

http://www.learner.org/courses/learningmath/measurement/session1/part_a/index.html)

The beginning of this passage appears circular, since saying that a property "can be quantified using some kind of unit as a basis" just describes what we do when we measure. So the first sentence is just saying that a property is measurable if we can measure it. The remainder of the first paragraph and the last paragraph get closer to the essentials, in referring to counting units and in saying that measurable attributes "increase when the objects are combined." But it is not enough that they simply increase. For example "variety of hair styles" increases when two crowds of people are combined, but "variety of hair styles" is not a measurable attribute of a crowd, since there is no unit.

The task posed here is to write and improved explanation of "What is a measurable property, and how do we recognize one?" using the ideas attributed to Euclid in section 1.1.

6. In the same workshop (homework for Session 1), the following problem is posed:

> This table shows the height to the shoulder and the weight of several species of buffalo. Which is the largest (i.e., one that is both tall and heavy)? The guar is the tallest, and the water buffalo and the American bison are the heaviest, but how might we determine which is the largest?

Name (Country/Continent)	**Height (cm)**	**Weight (kg)**
Water Buffalo (Asia)	165	1,000
African Buffalo	135	560
Yak (Tibet)	200	550
Guar (Asia)	220	850
American Bison (North America)	180	1,000
European Bison	200	900

> Maria decides to add the height and the weight as a measure of the total size. The animal with the greatest sum is the largest. Which three animals, in order from largest to smallest, are the largest, according to Maria's criterion?

Is "total size" a measurable attribute of animals? (If you think so, what is the unit by which that attribute is measured?) Does this problem reinforce the basic ideas that are involved in measurement?