What is a polynomial? Motivating the university-level definition.

The Wikipedia answer is ...

A polynomial is "an expression of finite length constructed from variables and constants using only the operations of addition, subtraction, multiplication and non-negative integer exponents."

We can simplify this somewhat. If we can multiply by constants, we can multiply by -1. Multiplying a quantity by -1 and then adding it is the same thing as subtracting it. So, if we can multiply by constants and can add, then we can subtract. Therefore, we need not mention subtraction among the operations needed in order to make polynomials.

Another simplification relates to exponents. Integer exponents simply represent repeated multiplication, so with multiplication alone we can express anything that we might have written with exponents, e.g.,

$$a\,x^5 + b\,x^3 + x = a\,x\,x\,x\,x\,x\;\; + b\,x\,x\,x + x.$$

The following, then, is a more concise answer to our question: a polynomial is an expression of finite length constructed from variables and constants using only the operations of addition and multiplication.

But there is a problem with this answer.

The problem is this: *different expressions* may denote the *same polynomial*. For example,

$$(x+2)(x+3) \qquad \text{and} \qquad x^2 + 5x + 6 \tag{1}$$

are obviously different expressions, yet we regard them as the same polynomial, for we write,

$$(x+2)(x+3) = x^2 + 5x + 6. \tag{2}$$

A polynomial cannot simply be an expression, since different expressions may be the same as polynomials.

An interlude about fractions

The issue we face here is not unlike the issue we face when we try to say exactly what a fraction is. Is it correct to say that a fraction an expression consisting of a numerator and a denominator above and below a fraction bar? If this is what a fraction is, then

$$\frac{2}{4} \qquad \text{and} \qquad \frac{1}{2} \tag{3}$$

are different fractions, because they have different parts. But we write

$$\frac{2}{4} = \frac{1}{2}. \tag{4}$$

A fraction cannot simply be an expression, since different expressions may be the same as fractions.

The word "fraction" is actually used in two ways—sometimes to mean a *fractional expression* and sometimes to mean a *rational number*. A fractional expression is a configuration of symbols that is used to name a rational number. Many different fractional expressions name the same rational number, and two fractional expressions name the same rational number if they can be transformed into one another by cancelling or "un-cancelling" common factors from numerator and denominator—or if we can transform both to the same *reduced* fractional expression by cancelling all common factors from the numerator and denominator. Note in particular that every rational number has a favored name consisting of a reduced fractional expression, and that there is a procedure that transforms any given name into the favored name.

Problem 1. Two fractional expressions might refer to the same rational number, but it might not be obvious that they do so. Which of the following denote the same number?

a) 1034859/2135631

b) 1064768/2197354

c) 1031232/2128146

d) 1018248/2101351

Back to polynomials

Let us return to the question, "What is a polynomial?" Our comments about fractions suggest that we should distinguish between *polynomial expressions* and something else, more abstract, that would fill in the following analogy:

$$\text{fractional expressions : rational numbers :: polynomial expressions : } \boxed{\ ?\ }.$$

What goes in the box ought to be a kind of entity on a par with rational numbers—a class of things that are not literal polynomial expressions, but that can be referred to by using polynomial expressions. This is what polynomials ought to be.

This might not seem like a complete answer to the original question, "What is a polynomial?", but if we examine the implications we will see that this provides important information. A polynomial expression is a configuration of symbols that is used to name a polynomial. Many different polynomial expressions name the same polynomial, and two polynomial expressions name the same polynomial if they can be transformed into one another by using the laws of arithmetic (the commutative, associative and distributive laws as well as the facts of arithmetic with constants).

There is an easy, algorithmic way to check if two polynomial expressions can be transformed into one another. Check if both transform to the same *fully expanded* polynomial expression by using the distributive law repeatedly to expand all multiplications. Note in particular that every polynomial has a favored name consisting of a fully expanded polynomial expression written as a sum of monomial terms, and that there is a procedure that transforms any given name into the favored name.

For example, the two expressions in line (1) refer to the same polynomial because we can transform one to the other using the distributive and commutative laws:

$$\begin{aligned}(x+2)(x+3) &= (x+2)x + (x+2)3 \\ &= (x+2)x + 3(x+2) \\ &= xx + 2x + 3x + 6 \\ &= x^2 + 5x + 6.\end{aligned}$$

The expression on the last line is fully expanded.

Problem 2. Two polynomials might be the same, but *not obviously* the same—or, to say this more precisely, two polynomial expressions might refer to the same polynomial, but it might not be obvious that they do so. Which of the following denote the same polynomial?

a) $p(x) = ((105x - 1033)x + 2911)x - 2431$
b) $q(x) = (3x - 17)(35x^2 - 146x + 143)$
c) $r(x) = (7x - 11)(15x^2 - 116x + 221)$
c) $s(x) = 105x^3 - 997x^2 + 2823x - 2431$

Standard form

It is a fact that by repeated use of the distributive law, one can always reduce a polynomial to a sum of monomial terms:

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, \quad a_0, a_1, \ldots, a_n \in \mathbb{R},\ a_n \neq 0.$$

This is called "standard form." The existence of standard form provides a simple procedure to check to see if two polynomial expressions refer to the same polynomial: simply reduce both to standard form and check to see if the coefficients are the same. Or take the difference, and reduce it to standard form. The standard form of the difference of two polynomial expressions is 0 if and only if the two expressions denote the same polynomial.

Exercise 1. Prove the fact asserted in the previous paragraph by induction on the number of alternations between multiplication and addition. (Begin by defining alternations.)

Note that sometimes we insist that the terms are written in order of decreasing degree, and sometimes some other order is chosen. Once a polynomial has been fully expanded, i.e., written as a sum of monomial terms, it is very easy to rearrange the terms in whatever order is desired. In order to record all the information in the standard form, we just need to remember what coefficients go with what powers of x. So, we can code a polynomial in standard form by its coefficients:

$$a_0 + a_1\, x + \cdots + a_n\, x^n \quad \Leftrightarrow \quad (a_0, a_1, \cdots, a_n).$$

If $\mathbb{F}$ is a field then the set of polynomials with coefficients from $\mathbb{F}$ is denoted $\mathbb{F}[x]$.

Are polynomials functions?

Any algebraic expression is a recipe for a function—to evaluate the function at a particular number, substitute the number for the variable and simplify. A function that is defined by a polynomial expression is called a *polynomial function*.

As we know, a function is not fully described until its domain is specified. In the case of the polynomial functions that one sees in high school, the domain is usually assumed to be $\mathbb{R}$, the field of real numbers. But a polynomial expression with coefficients from $\mathbb{R}$ can also be evaluated when a complex number is substituted for the variable, so we could assume the domain to be $\mathbb{C}$. Generally when working with functions, we assume the largest possible domain unless a smaller domain is specified. A polynomial with coefficients from $\mathbb{R}$ can be evaluated at any number in any field that contains $\mathbb{R}$, and $\mathbb{C}$ is not the largest of these. In fact, there is no largest field containing $\mathbb{R}$, so there is no optimal choice of domain.

Our discussion of "What is a polynomial?" concluded that polynomials are equivalence classes of expressions. Provided that a sufficiently large domain is chosen, it is in fact the case that two polynomial expressions (in a single variable) denote the same polynomial (i.e., have the same standard form) if and only if they define the same function (on the chosen domain). How big must the domain be? It is enough to demand that it have infinitely many elements.

You may be aware of the existence of finite fields $\mathbb{F}_p$. The polynomials with coefficients from $\mathbb{F}_p$ can *not* be identified with functions from $\mathbb{F}_p$ to $\mathbb{F}_p$. There are only finitely many such functions, but $\mathbb{F}_p[x]$ is infinite. However, $\mathbb{F}_p$ is contained in an infinite field (e.g., the algebraic closure of $\mathbb{F}_p$, or any transcendental extension of $\mathbb{F}_p$), and if we view a polynomial in $\mathbb{F}_p[x]$ as a function on such a domain, then different polynomials give different functions.

So, what is the final answer? What is a polynomial?

A polynomial is an equivalence class of polynomial expressions. Two polynomial expressions denote the same polynomial if and only if they have the same standard form, and this is the case if and only if the expressions define the same polynomial function on a sufficiently large domain.

Answers

Problem 1.

a) $\frac{1034859}{2135631} = \frac{3\times 7\times 49279}{3\times 711877} = \frac{7\times 49279}{711877}$,

b) $\frac{1064768}{2197354} = \frac{2^6\times 127\times 131}{2\times 41\times 127\times 211} = \frac{2^5\times 131}{41\times 211}$,

c) $\frac{1031232}{2128146} = \frac{2^6\times 3\times 41\times 131}{2\times 3\times 41^2\times 211} = \frac{2^5\times 131}{41\times 211}$,

d) $\frac{1018248}{2101351} = \frac{2^3\times 3\times 7\times 11\times 19\times 29}{7\times 300193} = \frac{2^3\times 3\times 11\times 19\times 29}{300193}$.

Problem 2.

a) $p(x) = ((105x - 1033)x + 2911)x - 2431 = -2431 + 2911x - 1033x^2 + 105x^3$

b) $q(x) = (3x - 17)(35x^2 - 146x + 143) = -2431 + 2911x - 1033x^2 + 105x^3$

c) $r(x) = (7x - 11)(15x^2 - 116x + 221) = -2431 + 2823x - 997x^2 + 105x^3$

c) $s(x) = 105x^3 - 997x^2 + 2823x - 2431 = -2431 + 2823x - 997x^2 + 105x^3$