**M7210 Lecture 4** Monday August 27, 2012

**Vector Spaces: Definition and Examples**

I will assume familiarity with matrix multiplication, with row reduction and row-echelon form and with the manner in which a system of linear equations may be represented by an augmented matrix and solved by row reduction. (See Chapter 1 of the textbook for a review.)

Let $\mathbb{F}$ be a field (e.g., $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$).

**Definition.** A *vector space* over $\mathbb{F}$ is a set $V$ equipped with:

a) a binary operation $+$ on $V$, a unary operation $-$ on $V$ and a constant $0_V \in V$, which together make $(V, +, -, 0_V)$ into an abelian group;

b) a *scalar multiplication* with coefficients from $\mathbb{F}$, i.e., a way of multiplying elements of $V$ by elements of $\mathbb{F}$, with the properties that for all $a, b \in \mathbb{F}$ and $v, w \in V$:
   i. $1_{\mathbb{F}} v = v$;
   i. $a(b\,v) = (a\,b)\,v$;
   ii. $(a + b)\,v = (a\,v) + (b\,v)$;
   iii. $a(v + w) = (a\,v) + (a\,w)$.

Many properties follow. For example, $0_{\mathbb{F}} v = 0_V = a\,0_V$ for all $v \in V$ and $a \in \mathbb{F}$. Also, $(-1_{\mathbb{F}})v = -v$ for all $v \in V$.

*Examples.* I will mention some important constructions. (For more, see the textbook.)

1) $\mathbb{F}^n$ denotes the set of $n$-tuples of elements of $\mathbb{F}$. It is a vector space with component-wise operations. The elements

$$e_1 = (1, 0, \ldots, 0)\,,\ e_2 = (0, 1, \ldots, 0)\,, \ldots,\ e_n = (0, 0, \ldots, 1)$$

are called the *standard basis* for $\mathbb{F}^n$. Every element of $\mathbb{F}^n$ has an expression as a linear combination of the $e_i$ that is unique up to the order of the summands:

$$(a_1, a_2, \ldots, a_n) = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n.$$

2) Let $S$ be any set, and let $S^{\mathbb{F}}$ be the set of functions from $S$ to $\mathbb{F}$. Addition and scalar multiplication operate "pointwise"—that is, for any $f, g \in S^{\mathbb{F}}$ and $s \in S$,

$$(f + g)[s] = f[s] + g[s] \qquad (af)[s] = af[s].$$

(Here, I have used square braces to indicate the argument of a function.) If $S$ is finite, then the family of functions $\{\, \delta_s \mid s \in S \,\}$, where $\delta_s$ defined by

$$\delta_s[t] := \begin{cases} 1, & \text{if } s = t; \\ 0, & \text{if } s \neq t, \end{cases}$$

has the property that every element of $\mathbb{F}^S$ can be written uniquely as a linear combination of the $\delta_s$. Indeed, when $S$ is finite and $f \in \mathbb{F}^S$, then we have:

$$\text{for all } t \in S,\ f[t] = \sum_{s \in S} f[s]\delta_s[t].$$

When $S$ is infinite, there is no easy way to identify a basis for $\mathbb{F}^S$.

3) Let $E$ be any set. Consider the expressions $a_1e_1 + a_2e_2 + \cdots + a_ke_k$, where $k \in \mathbb{N}$, $a_1, \ldots, a_k \in \mathbb{F}$, $e_1, \ldots, e_k \in E$. Let us agree to count two such expressions equivalent if one can be converted to the other using the algebraic rules in the definition above. For example, if $E = \{r, s, t\}$, then $r + 2s + 3t + 4r + 5s - t - t - t$ is equivalent to $5r + 7s$. Let $\bigoplus_E \mathbb{F}$ denote the set of all equivalence classes of expressions. This is called the *free $\mathbb{F}$-vector space on $E$*. If we order $E$, then in each equivalence class, there is a unique "fully simplified" representative, obtained by combining like terms, omitting zero terms and then arranging the non-zero terms according to the chosen order on $E$. We may identify $\bigoplus_E \mathbb{F}$ with the set of all fully simplified expressions. We add two expressions by writing a plus sign between them and then simplifying. Scalars operate by distributing over terms.
4) If $V$ is a vector space and $U$ is a subset of $V$ that is closed under addition and scalar multiplication, then $U$ is a vector space. $U$ is said to be a *sub-vector-space* of $V$.

## 1. Spanning, Independence, Bases and Dimension

The main goal of this section is to define the dimension of a vector space (in the finite-dimensional case). In pursuit of this goal, we define and study the other concepts in the title.

**Definition.** Let $S = \{s_1, s_2, \ldots,\} \subseteq V$. We say $S$ *spans* $V$ if every element of $V$ is a linear combination of elements of $S$. We say $S$ is *independent* if no linear combination of elements of $S$ is equal to $0_V$ except the one with all coefficients equal to $0_{\mathbb{F}}$. We say $S$ is a basis if it is independent and it spans $V$.

In Examples 1, 2 and 3 in the previous section, we exhibited some special bases for some special vector spaces.

**Proposition 1.1.** *Suppose $V$ has a basis $\{v_1, \ldots, v_n\}$. Then for each element $v \in V$, there is a unique $n$-tuple $(c_1, \ldots, c_n) \in \mathbb{F}^n$ such that*

$$v = c_1\, v_1 + c_2\, v_2 + \cdots + c_n\, v_n.$$

**Homework Exercise 1** is to write out the proof in detail. It is a direct consequence of the definition.

**Proposition 1.2.** *a) A minimal spanning set is independent. b) A maximal independent subset spans.*

*Proof.* a) If $S$ is not independent, then some element of $S$ can be written as a linear combination of others, so it can be omitted without destroying the spanning property. b)

If $S$ does not span, then some element of $V$ cannot be written as a linear combination of elements of $S$, so it can be added to $S$ without destroying independence. /////

**Homework Exercise 2.** Suppose $V$ is a sub-vector-space of $W$ . Show that a basis for $V$ may be extended to a basis for $W$. Give an example showing that a basis for $W$ might not contain a basis for $V$.

**Proposition 1.3.** *If $V$ has a spanning set $\{s_1, \ldots, s_n\}$ with $n$ elements, then every independent subset of $V$ has $\leq n$ elements.*

*Proof.* We show that any subset of $V$ with $m = n + 1$ elements is not independent. Let

$$\begin{aligned} v_1 &= c_{11}s_1 + \cdots + c_{1n}s_n \\ v_2 &= c_{21}s_1 + \cdots + c_{2n}s_n \\ &\vdots \\ v_m &= c_{m1}s_1 + \cdots + c_{mn}s_n. \end{aligned}$$

We claim that we can find $(x_1, \ldots, x_m) \in \mathbb{F}^{n+1}$ different from $(0, \ldots, 0)$ so that

$$x_1v_1 + \ldots + x_mv_m = 0_V. \tag{$*$}$$

Consider the system of $n$ homogeneous equations in $m = n + 1$ variables represented by the following table, with the columns corresponding to the $x_i$.

$$\begin{array}{cccc|c} c_{11} & c_{21} & \ldots & c_{m1} & 0 \\ c_{12} & c_{22} & \ldots & c_{m2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ c_{1n} & c_{2n} & \ldots & c_{mn} & 0 \end{array}$$

Since are more variables than equations, there is a non-zero solution, and this satisfies $(*)$. /////

**Proposition 1.4.** *If $V$ has a finite spanning set, then it has a finite basis, and every basis of $V$ has the same number of elements.*

*Proof*. The spanning set contains a basis by Proposition 1. If there is a basis of cardinality $n$ and a base of cardinality $n'$ then since both sets are both spanning sets and independent, by Proposition 2, $n \leq n'$ and $n' \leq n$. /////

**Definition.** If $V$ has a finite spanning set, the number of elements in a basis for $V$ is called the dimension of $V$ and is denoted $\dim V$.

**Proposition 1.5.** *Suppose $V$ is finite-dimensional. If $U$ is a sub-vector-space of $V$ , then $U$ is finite dimensional and $\dim U \leq \dim V$. If $U \subseteq V$ and $\dim U = \dim V$, then $U = V$.*

**Homework Exercise 3.** Prove Proposition 1.5.

Later, we will extend the concept of dimension to vector spaces that do not have finite spanning sets.