M7210 Lecture 5. Vector Spaces Part 2, Matrices Wednesday August 29, 2012
[delayed due to Hurricane Isaac]

In the present section, we look at how matrices are related to linear maps between vector spaces of the form $\mathbb{F}^n$ and with vector spaces that arise in this context. The conventions of matrix multiplication require us to use column vectors (rather than row vectors) to denote elements of $\mathbb{F}^n$.

An $m \times n$ matrix is written as follows:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \vdots & \vdots & \\ a_{m1} & a_{k2} & \dots & a_{mn} \end{pmatrix}$$

We will represent the elements $x \in \mathbb{F}^n$ and $y \in \mathbb{F}^m$ as columns:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} , \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} .$$

A gives rise to a function $L_A : \mathbb{F}^n \to \mathbb{F}^m$ that is defined by matrix multiplication:

$$y = L_A(x) = A\,x\,, \text{ with } y \text{ determined by } y_i = \sum_{j=1}^{n} a_{ij} x_j\,.$$

Note that by the properties of matrix multiplication, $L_A(ax + a'x') = aAx + a'Ax'$ for all $a, a' \in \mathbb{F}$ and all $x, x' \in \mathbb{F}^n$. Another consequence of the properties of matrix multiplication is the following:

Exercise. Show that if B is an $\ell \times m$ matrix, then $L_B \circ L_A = L_{BA}$.

$\mathbb{F}^n$ has a basis $\{e_1, \dots, e_n\}$ defined as follows:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} , \; e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} , \; \dots , \; e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} .$$

The image of e_j is $L(e_j) = A\,e_j =$ the j^{th} column of A. Any linear combination of the columns of A can be written as a matrix product $A\,b$, where $b \in \mathbb{F}^n$ records the coefficients.

Definition.

i) *column space of* A = span of the columns of $A = \{\, A\,b \mid b \in \mathbb{F}^n \} \subseteq \mathbb{F}^m$.

ii) *null space of* $A = \{x \mid A\,x = 0\,\} \subseteq \mathbb{F}^n$.

Theorem 2.1. *The dimension of the null space of A plus the dimension of the column space of A is the number of columns of A.*

Proof. Let $\{v_1, \ldots, v_r\} \subset \mathbb{F}^n$ be a basis for the null space of A. We may extend this set to a basis of F^n by selecting $v_i \in \mathbb{F}^n$ successively for $i = r+1, \ldots, n$ so that $\{v_1, \ldots, v_r, v_{r+1}, \ldots, v_i\}$ is independent. The column space is the span of $\{\, A\,v_1, \ldots, A\,v_n \,\}$, but $A\,v_1 = \cdots = A\,v_r = 0$, so the column space is the span of $\{\, A\,v_{r+1}, \ldots, A\,v_n \,\}$. It remains to show that $\{\, A\,v_{r+1}, \ldots, A\,v_n \,\}$ are independent. Suppose we can find $c_{r+1}, \ldots, c_n$ so that:

$$c_{r+1}\,A\,v_{r+1} + \ldots + c_n\,A\,v_n = 0.$$

We must show that $c_{r+1}, \ldots, c_n$ all vanish. Now, $c_{r+1}\,v_{r+1} + \ldots + c_n\,v_n$ is in the null space, so we may find $c_1, \ldots, c_r$ so that

$$c_1\,v_1 + \ldots + c_r\,v_r + c_{r+1}\,v_{r+1} + \ldots + c_n\,v_n = 0.$$

But $v_1, \ldots, v_n$ is a basis, so all the c_i must vanish. In particular, $c_{r+1}, \ldots, c_n$ vanish. /////

Homework. Do the Exercise in the lecture. Read pp. 40–41. Do page 82: 4, 5.