M7210 Lecture 7. Vector Spaces Part 4, Dual Spaces Wednesday September 5, 2012

Assume V is an n-dimensional vector space over a field $\mathbb{F}$.

Definition. The *dual space of* V, denoted V' is the set of all linear maps from V to $\mathbb{F}$.

Comment. $\mathbb{F}$ is an ambiguous object. It may be regarded:

(a) as a field,
(b) vector space over $\mathbb{F}$,
(c) as a vector space equipped with basis, $\{1\}$.

It is always important to bear in mind which of the three objects we are thinking about. In the definition of V', we use (c). The distinctions are often pushed into the background, but let us keep them in clear view for the remainder of this paragraph. Suppose V has basis $E = \{e_1, e_2, \ldots, e_n\}$. If $w \in V'$, then for each e_i, there is a unique scalar $\omega(e_i)$ such that $w(e_i) = \omega(e_i)\,1$. Then

$$\begin{pmatrix} w \\ \{1\}E \end{pmatrix} = \begin{pmatrix} \omega(e_1) & \omega(e_2) & \cdots & \omega(e_n) \end{pmatrix}. \tag{1}$$

The columns (of height 1) on the right hand side are the coefficients required to write the images of the basis elements in E in terms of the basis $\{1\}$. (The definition of the matrix of a linear map that we gave in the last lecture demands that each entry in the matrix be an element of the scalar field, not of a vector space.) The notation $w(e_i) = \omega(e_i)\,1$ is a reminder that $w(e_i)$ is an element of the *vector space* $\mathbb{F}$, while $\omega(e_i)$ is an element of the *field* $\mathbb{F}$. But scalar multiplication of elements of the *vector space* $\mathbb{F}$ by elements of the *field* $\mathbb{F}$ *is* good old fashioned multiplication $* : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$. Accordingly, as long as we are dealing with $\mathbb{F}$ in sense (c), we can assume $w(e_i) = \omega(e_i)$.*

Lemma. *Let V and E be as above. The map $C_E : V' \to \mathbb{F}^n$ defined by*

$$C_E(w) = \big(w(e_1), w(e_2), \cdots, w(e_n)\big).$$

Proof. We show C_E is injective. Suppose $C_E(w) = C_E(z)$ for some $w, z \in V'$. Then $w(e_i) = z(e_i)$ for $i = 1, \ldots, n$. We need to show $w = v$, i.e., $w(v) = z(v)$ for all $v \in V$. Let $v = \sum_{i=1}^n a_i e_i$ be any element of V. Then $w(v) = \sum_{i=1}^n a_i w(e_i)) = \sum_{i=1}^n a_i z(e_i)) = z(v)$, so the function is injective. We show the function is surjective. Let $(b_1, \ldots, b_n) \in \mathbb{F}^n$. We need to find $w \in V'$ such that $C_E(w) = (b_1, \ldots, b_n)$. By the universal mapping property of bases, there is a unique linear map w from V to $\mathbb{F}$ such that $w(e_i) = b_i$. Thus, the map is surjective (and hence bijective). Finally, we need to show that C_E is linear. Suppose $w, z \in V'$ and $a, b \in \mathbb{F}$. Then $(aw+bz)(e_i) = aw(e_i)+bz(e_i)$ by definition of function addition. Thus, $C_E(aw + bz) = aC_E(w) + bC_E(z)$. /////

Comment. We see that V' is a vector space of dimension equal to the dimension of V. The map that identifies V' with $\mathbb{F}^n$ depends on the choice of basis for V.

Comment. It is natural to regard the elements of V' as row vectors, as in equation (1). We have been regarding the elements of V as column vectors. A row of length n is a $1 \times n$ matrix that acts on an $n \times 1$ column by matrix multiplication.

The standard basis for $\mathbb{F}^n$ (viewed as a row space) consists of the row vectors δ_i, $i = 1, \ldots, n$. For each i, all the entries of δ_i are 0 except in the i-th coordinate, where it has a 1:

$$\delta_{ij} = (\delta_i)_j = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

* Later in this course, we will encounter the idea of a *forgetful functor*. This is an abstract device that enables us to deal with distinctions like those we have been making here in an efficient manner in instances when they are really important.

Suppose $e_i \in E$. We define e'_i by stipulating its value at each $e_j \in E$ as follows:

$$e'_i(e_j) = \delta_{ij}.$$

Clearly, any $w \in V'$ can be written as a linear combination of the e'_i and also clearly there are no non-trivial linear relations among the e'_i. Thus, the e'_i are a basis for V'. We call it E', the dual basis of E. Note that $C_E(e'_i) = \delta_i$.

Geometric/Algebraic Perspectives

Mathematicians intuitively view some objects as geometric and others as algebraic. For example, in analytic geometry we treat the plane $\mathbb{R}^2$ as a geometric object and we look at the subsets of $\mathbb{R}^2$ defined by the vanishing of one or more functions, e.g., $\{\,(x, y) \mid f(x, y) = 0 \,\&\, g(x, y) = 0\,\}$. On the other hand, the polynomials themselves may be added and multiplied, and the set of all of them forms a ring $\mathbb{R}[x, y]$.

Even though V and V' are both vector spaces of the same dimension (assuming finite dimension), it is sometimes useful to think of V as a geometric object in which we might define subsets by the vanishing of linear maps $w : V \to \mathbb{F}$ and to think of V' as an algebraic object.[2]

Definition. If U is a subspace of V, then $\operatorname{Ann}(U) := \{w \in V' \mid w(u) = 0 \text{ for all } u \in U\,\}$.

Proposition. *If V is finite-dimensional and U is a subspace of V,*
(a) $\dim U + \dim \operatorname{Ann}(U) = \dim V$.
(b) If $f \in U'$, then there is $g \in V'$ such that $g|_U = f$.
(c) If $y \in V \setminus U$, there is $g \in \operatorname{Ann}(U)$ such that $g(y) = 1$.

Proof. (Sketch) (a) Let $\{\,v_1, \ldots, v_n\,\}$ be a basis of V with the first r elements being a basis of U. Then $v'_{r+1}, \ldots, v'_n$ are contained in $\operatorname{Ann} U$ (why?) and they span $\operatorname{Ann} U$ (why?). Hence they form a basis for $\operatorname{Ann}(U)$ (why?). (b) $v'_1|_U, \ldots, v'_r|_U$ span U' (why?). So every element of U' is of the form $c_1v'_1|_U + \ldots + c_rv'_r|_U = \big(c_1v'_1 + \ldots + c_rv'_r\big)|_U$. (c) We can construct $\{\,v_1, \ldots, v_n\,\}$ so that $v_{r+1} = y$.

Exercise. If S is a subset of V, define

$$\operatorname{Ann}(S) := \{w \in V' \mid w(s) = 0 \text{ for all } s \in S\,\}.$$

(This generalizes the definition above to subsets.) If T is a subset of V', define

$$Z(T) := \{v \in V \mid t(v) = 0 \text{ for all } t \in T\,\}.$$

Show:
a. $\operatorname{Ann}(S)$ is a subspace of V'.
b. $Z(T)$ is a subspace of V.
c. $Z(\operatorname{Ann}(S))$ is a subspace of V spanned by S.

[2] One place where we do exactly this is in defining the tangent space and the cotangent space of a real manifold at a point P. The tangent space plays a geometric role, representing the linear structure of infinitesimal piece of the manifold about P. The cotangent space plays an algebraic role, representing the linear part of the germs at P of the smooth $\mathbb{R}$-valued functions that vanish at P. The cotangent space at P is the dual of the tangent space at P.