M7210 Lecture 8. Vector Spaces Part 5 Friday September 7, 2012

Double dual

If V is finite-dimensional, then V and V' have the same dimension, so there is a linear bijection between them. In fact, there are many, and there is no way (if no basis is specified) of picking out one as special.

But now consider V'', the dual of the dual of V. If $z \in V''$, then z is a linear map $z : V' \to \mathbb{F}$. Notice that each element of $v \in V$ determines a linear map from V' to $\mathbb{F}$ as follows: in the expression $w(v)$ with $w \in V'$, view v as fixed and view w as the variable. This is indeed linear, because $(aw_1 + bw_2)(v) = a(w_1(v) + b(w_2(v))$, by the definition of addition and scalar multiplication of functions. We need a name for this function. We call it $\iota(v) \in V''$. The rule by which is acts on $w \in V'$ is:

$$\iota(v)\big(w\big) := w(v).$$

Proposition. *If V is finite-dimensional, $\iota : V \to V''$ is a linear bijection.*

Proof. We know that V and V'' have the same dimension, so it is enough to show that ι is injective. It suffices to show that if $\iota(v) = 0$, then $v = 0$. But $\iota(v) = 0$ means $w(v) = 0$ for all $w \in V'$, and this can happen only if $v = 0$. /////

Notice that the map ι was defined without reference to any basis.

Exercise 1.

(a) Let $E = \{e_1, \ldots, e_n\}$ be a basis for V. Let E' be the dual basis (as defined in the last lecture), and let $E'' = (E')'$ be the dual basis of E'. Show that $\iota(e_i) = e_i''$.

(b) If $L : V \to W$ is any linear map between finite-dimensional vectors spaces, then we may define a linear map $L'' : V'' \to W''$ by stipulating $L''(z) = \iota(L(\iota^{-1}(z)))$. Prove that L'' is a linear map.

Optional Project: Duality of Convex Cones. Let V be a finite-dimensional vector space over $\mathbb{R}$. A subset C of V is said to be a *homogeneous convex cone*—or an *HCC*—if $au + bv \in C$ whenever $u, v \in C$ and $a, b \in \mathbb{R}_{\geq 0}$. We say C is a *polyhedral HCC*—or *PHCC*—if there is a finite set $v_1, \ldots, v_k \in V$ such that

$$C = \{\, a_1 v_1 + \ldots + a_k v_k \mid a_1, \ldots, a_k \in \mathbb{R}_{\geq 0} \,\}.$$

If $w \in V'$, we define the *non-positive set of* w to be $N(w) = \{\, v \in V \mid w(v) \leq 0 \,\}$. If C is an HCC, the *dual cone* of C is

$$C' := \{\, w \in V' \mid C \subseteq N(w) \,\}.$$

(a) Show that C' is an HCC in V'.

(b) (*) Show that if C is a PHCC, then so is C'.

(c) (*) Show that if C is a PHCC, then $\iota(C) = C''$.

The symbol (*) is a warning that (b) and (c) are hard. An excellent resource for this topic is: Harold W. Kuhn and Albert William Tucker, *Linear Inequalities and Related Systems*, Princeton, 1956. Convexity theory has a lot of practical applications (linear programming). In the 1990s, convex cones became very important in algebraic geometry; see William Fulton, *Introduction to Toric Varieties*, Princeton,1993.

Quotient Spaces

We are going to examine quotients of vector spaces as an example of a far more general type of construction that occurs throughout algebra.

Quotients "in the category of sets"

Let X be a set and let R be a relation $R \subseteq X \times X$. We write xRy to mean $(x, y) \in R$.

R is said to be an *equivalence relation* if it is reflexive (i.e., xRx for all $x \in X$), symmetric (i.e., $xRy \Rightarrow yRx$ for all $x, y \in X$) and transitive (i.e., $xRy \,\&\, yRz \Rightarrow yRz$ for all $x, y, z \in X$).

Assume R is an equivalence relation on X. The set $[x]_R := \{\, y \in X \mid xRy \,\}$ is called the *equivalence class of* x. For any $x, y \in X$, either $[x]_R = [y]_R$ or $[x]_R \bigcap [y]_R = \emptyset$. The set of equivalence classes is denoted X/R.

If $f : X \to Y$ is any function of sets, then $K_f := \{\, (x, y) \mid f(x) = f(y) \,\}$ is an equivalence relation on X and we have a factorization $f = ms$ where $s : X \to X/R; x \mapsto [x]_{K_f}$ is a surjective function and $m : X/R \to Y; [x]_{K_f} \mapsto f(x)$ is injective.

Quotients of vector spaces

Suppose V is a vector space with subspace U. We define a relation R_U on V by

$$v_1 R_U v_2 \Leftrightarrow v_1 - v_2 \in U.$$

Exercises.

(a) R_U is an equivalence relation on V.

(b) $[v]_{R_U} = \{\, v + u \mid u \in U \,\}$.

Remark: The set $[v]_{R_U}$ is more commonly denoted $v + U$. The set of equivalence classes is denoted V/U. Thus, $V/U = \{\, v + U \mid v \in V \,\}$.

(c) Show that for all $a, b \in \mathbb{F}$, $v_1, v_2, w_1, w_2 \in V$:

$$\text{if } v_1 - v_2 \in U \text{ and } w_1 - w_2 \in U, \text{ then } \Big(av_1 + bw_1\Big) - \Big(av_2 + bw_2\Big) \in U.$$

Remark: (c) implies that R_U is a subspace of $V \times V$. (The operations on $V \times V$ are "component-wise".)

(d) Using (c), show that the rules

$$(v + U) + (w + U) := (v + w) + U,$$

$$a(v + U) := (av) + U$$

are well-defined vector-space operations V/U. (The task is to show that the RHS does not depend on the particular representatives chosen to name the equivalence classes on the LHS.)

Remark: From (d), we conclude that V/U is a vector space under the operations defined.

(e) Suppose $L : V \to W$ is a linear map. Let $\ker L = \{\, v \in V \mid L(v) = 0 \,\}$. Show that $L = MS$, where $S : V \to V/\ker L$ is a surjective linear map, and $M : V/\ker L \to W$ is an injective linear map.