M7210 Lecture 9. UMPs for Products and Sums Monday September 10, 2012

Products

Let A and B be sets. The *product* of A and B is the set

$$A \times B := \{ (a,b) \mid a \in A,\, b \in B \}.$$

The product comes equipped with *projection functions* $\pi_A : A \times B \to A; (a,b) \mapsto a$ and $\pi_B : A \times B \to B; (a,b) \mapsto b$.[1]

The product together with the projections has the following *universal mapping property*:

UMP for $A \times B$. *If T is any set and $f_A : T \to A$ and $f_B : T \to B$ are any functions, then there is a unique function $F : T \to A \times B$ such that $\pi_A \circ F = f_A$ and $\pi_B \circ F = f_B$.*

Proof. Let $F(t) := (f_A(t), f_B(t))$. Then obviously $\pi_A \circ F = f_A$ and $\pi_B \circ F = f_B$. Suppose $G : T \to A \times B$ and $\pi_A \circ G = f_A$ and $\pi_B \circ G = f_B$. Then for all $t \in T$: $\pi_A(G(t)) = \pi_A(F(t))$ and $\pi_B(G(t)) = \pi_B(F(t))$. So $F = G$. /////

Lemma. *Suppose P is a set equipped with two functions $\phi_A : P \to A$ and $\phi_B : P \to B$ and suppose that P together with these functions has the following UMP: If U is any set and $g_A : U \to A$ and $g_B : U \to B$ are any functions, then there is a unique function $G : U \to P$ such that $\phi_A \circ G = g_A$ and $\phi_B \circ G = g_B$. Given all this data, there is a bijection $\Pi : A \times B \to P$ such that $p_A \circ \Pi = \pi_A$ and $p_B \circ \Pi = \pi_B$; moreover, Π is unique.*

Proof. We get Π by applying the UMP for P to $U = A \times B$, $g_A = \pi_A$, $g_B = \pi_B$. The uniqueness is guaranteed by the UMP, so all we need to do is show that Π is a bijection. Get Φ by applying the UMP for $A \times B$ to $T = P$, $f_A = \phi_A$, $f_B = \phi_B$. Then $\Phi \circ \Pi = \mathrm{id}_{A\times B}$ and $\Pi \circ \Phi = \mathrm{id}_P$, using the uniqueness clause in the UMPs to conclude that the identity is the only lifting of the projections; cf. footnote [1]. Since Pi has a two-sided inverse, it is a bijection. /////

Moral. Products are described uniquely by a UMP. Rather than using the set-theoretic construction to make the product, we could defined it "implicitly" by the UMP. The notation used to denote the solution of the UMP (if it exists) is $A \times B$, π_A and π_B.

Observe that the UMP is neutral concerning the kinds of object and maps.. The analogous UMP defines products of algebraic systems, be they vector spaces, groups, rings, etc.[2] For example, let U, V and W be vector spaces. If we have two vector-space maps $\phi_U : W \to U$ and $\phi_V : W \to V$ such that for any vector space T and any vector-space maps $f_U : T \to U$ and $f_V : T \to V$, there is a unique vector-space map $F : T \to W$ such that $\phi_U \circ F = f_U$ and $\phi_V \circ F = f_V$, then we say that W is a product of U and V. Since products are unique up to unique isomorphism, it is usual to say "the" instead of "a".

[1] Reminder: Ordered pairs are determined their entries. If $p_1 = (a_1, b_1)$ and $p_2 = (a_2, b_2)$ are two elements of $A \times B$, then

$$p_1 = p_2 \;\Leftrightarrow\; a_1 = a_2 \;\&\; b_1 = b_2 \;\Leftrightarrow\; \pi_A(p_1) = \pi_A(p_2) \;\&\; \pi_B(p_1) = \pi_B(p_2).$$

[2] One needs to be cautious—an implicit definition does not necessarily have a solution. For example, we could ask for a solution to the UMP for fields and maps of fields, but there is generally no solution.

Exercises.

(a) Suppose W is the product of U and V, as in the last paragraph. Show directly from the UMP, that the maps ϕ_U and ϕ_V must be surjective. (Hint. Consider $T = U$, and let $f_U = \mathrm{id}_U$ and $f_V = 0$.)

(b) Suppose that W contains subspaces U and V such that $U \cap V = \{0\}$ and $U \cup V$ spans W. Show that every element of W can be written uniquely in the form $u + v$, with $u \in U$ and $v \in V$. Show that there are maps $\pi_U : W \to U$, $\pi_V : W \to V$ making W into the product of U and V. (In this case, we refer to W as the "internal product of U and V".)

(c) Suppose U and V are vector spaces. Let $U \times V$ denote the set-theoretic product of U and V, with component-wise operations. (This is what we have naively been calling the product, all along.) Show that there are projection maps that make it into a vector-space product in the sense of the UMP. (Note that, U and V are not literally subspaces of $U \times V$. We call $U \times V$ the "external product of U and V".)

(d) State the UMP that characterizes $\prod\{\, S_\alpha \mid \alpha \in A \,\}$. Show that the set-theoretic product of sets $\{\, S_\alpha \mid \alpha \in A \,\}$ is the set of functions $f : A \to \bigcup\{\, S_\alpha \mid \alpha \in A \,\}$ with $f(\alpha) \in S_\alpha$ for all α. In particular, if S_α is the same set for all alpha, then the product is the set of all S-valued functions on A. (We denote this $\prod_A S$.)

(e) Let V be a vector space. What is the infinite vector-space product $\prod_A V$?

Sums

Let A and B be sets. We write $A \uplus B$ for the disjoint union of A and B. That is, we select copies $\tilde{A}$ and $\tilde{B}$ of A and B that are disjoint and set $A \uplus B = \tilde{A} \cup \tilde{B}$. Let $i_A : A \to A \uplus B; a \mapsto \tilde{a}$ and $i_B : B \to A \uplus B; b \mapsto \tilde{b}$ be the injections.

UMP for $A \uplus B$. *If T is any set and $h_A : A \to T$ and $h_B : B \to T$ are any functions, then there is a unique function $H : A \uplus B \to T$ such that $H \circ i_A = h_A$ and $H \circ i_B = h_B$.*

Proof. Let

$$H(\tilde{x}) := \begin{cases} h_A(x), & \text{if } \tilde{x} \in \tilde{A}; \\ h_B(x), & \text{if } \tilde{x} \in \tilde{B}. \end{cases} \qquad /////$$

Lemma. *Let A and B be sets, Suppose S is a set equipped with two functions $j_A : A \to S$ and $j_B : B \to S$ and suppose S together with these functions has the following UMP: If U is any set and $g_A : A \to U$ and $g_B : B \to U$ are any functions, then there is a unique function $G : S \to U$ such that $G \circ j_A = g_A$ and $G \circ j_B = g_B$. Given all this data, there is a bijection $I : S \to A \uplus B$ such that $I \circ j_A = i_A$ and $I \circ j_B = i_B$; moreover, I is unique.*

Proof. Exercise.

When we attempt to carry this construction over to algebraic systems, we run into a problem. Suppose U and V are vector spaces. It is not generally the case that $U \uplus V$ is a vector space. It has two zeroes, and it has no addition. It looks like the gig is up!

But no! It is actually the case that $U \times V$ with the injections $i_U : U \to U \times V; u \mapsto (u, 0)$ and $i_V : V \to U \times V; v \mapsto (0, v)$ solves the UMP for sums of vector spaces.

Exercise.

(a) Prove the claim after "But no!"

(b) Give an example to show that $H_1 \times H_2$ is NOT the sum of H_1 and H_2 in the sense above if we are working with groups that are not necessarily abelian. (Hint. Suppose T is a group that contains two elements a and b such that $ab \neq ba$. Let $H_1 = \mathbb{Z} = H_2$ and let $g_1(n) = a^n$ and $g_2(n) = b^n$. Then, we cannot define $G : H_1 \times H_2 \to T$ in a manner that satisfies the requirements of the UMP.