**M7210 Lecture 10.** Wednesday September 12, 2012

### Multilinear Functionals

Let $V$ be a vector space of dimension $n$ over $\mathbb{F}$. We will use bold letters $\mathbf{v}$ to stand for $k$-tuples of elements of $V$, i.e., elements of $V^k$.[1]

**Definition.** A *$k$-multilinear functional on $V$* is a function $f : V^k \to \mathbb{F}$ that is linear in each vector variable separately. I.e., for all $a, b \in \mathbb{F}$:

$$\begin{gathered} a\, f(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) + b\, f(\mathbf{v}'_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = f(a\mathbf{v}_1 + b\mathbf{v}'_1, \mathbf{v}_2, \dots, \mathbf{v}_k) \\ a\, f(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots \mathbf{v}_k) + b\, f(\mathbf{v}_1, \dots, \mathbf{v}'_i, \dots, \mathbf{v}_k) = f(\mathbf{v}_1, \dots, a\mathbf{v}_i + b\mathbf{v}'_i, \dots, \mathbf{v}_k) \\ \textit{etc.} \end{gathered}$$

In other words, if we hold all the vector variables in all places *except* the $i^{th}$ fixed, then the resulting function from $V$ to $\mathbb{F}$ is linear.

*Example.* Suppose $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is a basis for $V$ and suppose $a_{ij} \in \mathbb{F}$ for $i = 1, 2, 3$, $j = 1, 2$. Let

$$\begin{aligned} \mathbf{a}_1 &= a_{11}\varepsilon_1 + a_{12}\varepsilon_2 + a_{13}\varepsilon_3 \in V, \\ \mathbf{a}_2 &= a_{21}\varepsilon_1 + a_{22}\varepsilon_2 + a_{23}\varepsilon_3 \in V, \\ \mathbf{a} &= (\mathbf{a}_1, \mathbf{a}_2) \in V^2 \end{aligned}$$

If $f : V^2 \to \mathbb{F}$ is 2-multilinear, then

$$\begin{aligned} f(\mathbf{a}) &= f(\mathbf{a}_1, \mathbf{a}_2) \\ &= a_{11} f(\varepsilon_1, \mathbf{a}_2) + a_{12} f(\varepsilon_2, \mathbf{a}_2) + a_{13} f(\varepsilon_3, \mathbf{a}_2) \\ &= a_{11}\Big(a_{21} f(\varepsilon_1, \varepsilon_1) + a_{22} f(\varepsilon_1, \varepsilon_2) + a_{23} f(\varepsilon_1, \varepsilon_3)\Big) \\ &\qquad + a_{12}\Big(a_{21} f(\varepsilon_2, \varepsilon_1) + a_{22} f(\varepsilon_2, \varepsilon_2) + a_{23} f(\varepsilon_2, \varepsilon_3)\Big) \\ &\qquad\quad + a_{13}\Big(a_{21} f(\varepsilon_3, \varepsilon_1) + a_{22} f(\varepsilon_3, \varepsilon_2) + a_{23} f(\varepsilon_3, \varepsilon_3)\Big). \end{aligned}$$

Extending this pattern, we see that if $V$ has a basis $\mathcal{E} = \{\varepsilon_1, \dots, \varepsilon_n\}$, $f$ is $k$-multilinear and $\{a_{ij}\}$ is a $k \times n$ matrix with entries in $\mathbb{F}$, $\mathbf{a}_i = a_{i1}\varepsilon_1 + \cdots + a_{in}\varepsilon_n \in V$ and $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_k) \in V^k$, then

$$f(\mathbf{a}) = \sum_{\phi:[k]\to[n]} a_{1\phi(1)} \cdots a_{k\phi(k)} f(\varepsilon_{\phi(1)}, \dots, \varepsilon_{\phi(k)}). \tag{1}$$

Here, $[k] := \{1, 2, \dots, k\}$, $[n] := \{1, 2, \dots, n\}$, and we are summing over all possible functions.

**Exercise 1.** (a) Show that the set of all $k$-multilinear functionals on $V$ is a sub-vector-space of $F^{V^k}$ (the vector space of functions from $V^k$ to $\mathbb{F}$. (b) Show that if $f$ is a $k$-multilinear functional on $V$ and $L : V \to V$ is any linear function, then $f \circ (L, \dots, L)$ is $k$-multilinear, where $f \circ (L, \dots, L)(\mathbf{v}_1 \dots, \mathbf{v}_k) := f\big(L(\mathbf{v}_1), \dots, L(\mathbf{v_k})\big)$.

**Exercise 2.** (a) Let $\mathcal{E} = \{\varepsilon_1, \dots, \varepsilon_n\}$ be a basis of $V$. Suppose that $\omega : \mathcal{E}^k \to \mathbb{F}$ is any function. (In other words, $\omega$ is simply an assignment of elements of $\mathbb{F}$ to $k$-tuples of basis vectors.) Then, there is a unique $k$-multilinear functional $\overline{\omega} : V^k \to \mathbb{F}$ such that $\overline{\omega}|_{\mathcal{E}} = \omega$. (b) Show that the vector space of all $k$-multilinear functionals on $V$ has dimension $n^k$.

### Alternating multilinear functionals

**Definition.** A $k$-multilinear functional $f$ on $V$ is said to be *alternating* if $f(\mathbf{w}) = 0$ whenever $\mathbf{w} \in V^k$ has a repeated entry.

**Exercise 3.** Show that the set of all alternating $k$-multilinear functionals on $V$ is a vector space. Show that if $f$ is alternating, then so is $f \circ (L, \dots, L)$; cf. Exercise 1.

[1] Our textbook makes an effort to work with row vectors, and uses notation that reflects this. For example, the book uses $e_i^t$ to refer to the row associated with the column vector $e_i$ in the canonical basis of $\mathbb{F}^n$. I have decided to give a more abstract presentation, but have chosen my notation to remain compatible with the book. For example, I use $\varepsilon_i$ where the book might refer to $e_i^t$. (This whole footnote can be ignored, unless you want to look for the precise parallels between my notes and the book's treatment of this topic.)

**Lemma.** *Suppose $f$ is $k$-multilinear. Then $f$ is alternating if and only if*

$$f(\mathbf{v}) = -f(\mathbf{v}') \text{ whenever } \mathbf{v}' \in V^k \text{ is obtained from } \mathbf{v} \in V^k \text{ by transposing two entries.} \qquad (\tau)$$

*Proof*. Alternating implies $(\tau)$: Let $\widehat{f}$ be the functional that we obtain by holding fixed all arguments except those that are switched in passing from $\mathbf{v}$ to $\mathbf{v}'$ Then

$$\begin{aligned}
\widehat{f}(\mathbf{v}_1, \mathbf{v}_2) + \widehat{f}(\mathbf{v}_2, \mathbf{v}_1) &= \widehat{f}(\mathbf{v}_1, \mathbf{v}_2) + \widehat{f}(\mathbf{v}_1, \mathbf{v}_1) + \widehat{f}(\mathbf{v}_2, \mathbf{v}_1) + \widehat{f}(\mathbf{v}_2, \mathbf{v}_2) \\
&= \widehat{f}(\mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2) + \widehat{f}(\mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2) \\
&= \widehat{f}(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2) \\
&= 0.
\end{aligned}$$

Thus, $f(\mathbf{v}) + f(\mathbf{v}') = 0$, so $f(\mathbf{v}) = -f(\mathbf{v}')$.
$(\tau)$ implies alternating: **Exercise 4.** /////

**Determinants**

**Lemma 1.** *Suppose* $\dim V = n$. *The space of all alternating $n$-multilinear functionals on $V$ has dimension $\leq 1$.*

*Proof.* Let $\mathcal{E}$ be a basis for $V$. By the exercise, $f$ is determined by its restriction to $\mathcal{E}^n$. Since $f$ is alternating, it vanishes on any $n$-tuple in $\mathcal{E}^n$ with a repeated entry. Any $n$-tuple with no repeats is a permutation of $(\varepsilon_1, \ldots, \varepsilon_n)$, and that value of $f$ at any such element is equal to $\pm f(\varepsilon_1, \ldots, \varepsilon_n)$. /////

**Lemma 2.** *Suppose* $\dim V = n$. *There is a non-zero alternating $n$-multilinear functional on $V$.*

*Proof*. Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be a basis for $V$. Define a function $\mathcal{E}^n \to \mathbb{F}$ by setting $(\varepsilon_{\sigma(1)}, \ldots, \varepsilon_{\sigma(n)}) \mapsto \operatorname{sgn} \sigma$ for each permutation $\sigma$ of $\{1, \ldots, n\}$, and setting $(\varepsilon_{j(1)}, \ldots, \varepsilon_{j(n)}) \mapsto 0$ if $j$ is any function from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$ that is *not* injective. By Exercise 1, this defines a non-zero $n$-multlinear functional on $V$, which we call $D_{\mathcal{E}}$. If $\mathbf{a} \in V^k$ and $\{a_{ij}\}$ is the matrix of coefficients that we use to write $\mathbf{a}$ in terms of $\mathcal{E}$, as in Equation (1), then we have:

$$D_{\mathcal{E}}(\mathbf{a}) = \sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)} \operatorname{sgn} \sigma, \qquad (2)$$

where $S_n$ is the set of permutations of $[n]$ We need to show $D_{\mathcal{E}}$ it is alternating. So, suppose $\tau \in S_n$ is a transposition and $\mathbf{a}^\tau$ is defined by $(\mathbf{a}^\tau)_i = \mathbf{a}_{\tau(i)}$. Then

$$\begin{aligned}
D_{\mathcal{E}}(\mathbf{a}^\tau) &= \sum_{\sigma \in S_n} a_{\tau(1)\sigma(1)} \cdots a_{\tau(n)\sigma(n)} \operatorname{sgn} \sigma \\
&= \sum_{\sigma \in S_n} a_{1\sigma\tau(1)} \cdots a_{n\sigma\tau(n)} \operatorname{sgn} \sigma \qquad (\text{recall } \tau = \tau^{-1}) \\
&= \sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)} \operatorname{sgn} \sigma\tau \\
&= -\sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)} \operatorname{sgn} \sigma \\
&= -D_{\mathcal{E}}(\mathbf{a}). \qquad /////
\end{aligned}$$

*Comment.* In every one of the sums in the proof above, each of the products $a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{n\sigma(n)}$, $\sigma \in S_n$ occurs in exactly one term with a coefficient of either 1 or $-1$.

Note that we can define $D(A)$ for an arbitrary $n \times n$ matrix $A$ with entries from $\mathbb{F}$ using (2). We can interpret this as the case of (2) where $\mathcal{E}$ is the standard basis for $\mathbb{F}^n$, viewed as a row space.

**Homework.** Read the rest of the section on determinants. Do the four exercises in these notes. Also, attempt Problem 40, page 86.