M7210 Lecture 11. Friday September 14, 2012

Determinants (cont.)

Let us adjust notation to match the text. If A is an $n \times n$ matrix, the entry in its i^{th} row and j^{th} column will be written A_{ij}. Previously, I called this a_{ij}. So, all we are doing is using upper case letters where previously we had lower case. Also, in place of $D(A)$, we will write $\det A$. Both notations are widely used. When I refer to "the rows of A," I really mean the set of n tuple of elements of $\mathbb{F}^n$, where $\mathbb{F}^n$ is viewed as a row space.

In the last lecture, we showed:

Proposition 1. *The vector space of all alternating n-multilinear functional on the rows of A is* 1-*dimensional.* /////

We defined

$$\det A := \sum_{\sigma \in S_n} A_{1\sigma(1)} \cdots A_{n\sigma(n)} \operatorname{sgn} \sigma, \tag{10.2}$$

and showed that det is an non-zero alternating n-multilinear functional on the rows of A. By Proposition 1, det is—up to a constant multiple—the *only* n-multilinear functional on the rows of A. The following, which is obvious from the definition, tells us which multiple it is:

Proposition 2. *If I is the $n \times n$ identity matrix, then* $\det I = 1$. /////

Proposition 3. *If B is an $n \times n$ matrix, then* $\det(AB) = (\det A)(\det B)$.

Proof. Applying Exercise 10.1, $\det(AB)$ is an alternating linear function of the rows of A. By Proposition 1, $\det(AB) = \det A \, c(B)$, where $c(B)$ is a constant depending on B. If we let $A = I$, we get $\det B = c(B)$. /////

Proposition 4. $\det A = 0$ *if and only if A has no inverse.*

Proof. If A has an inverse, then $\det A$ has an inverse, so $\det A \neq 0$. If A does not have an inverse, then there is an invertible matrix B so that $B^{-1}AB$ has a row of zeroes, so $\det(B^{-1}AB) = 0$ and by Proposition 3, $\det A = 0$. ////

Proposition 5. *Let A^t be the transpose of A, i.e., $A^t_{ij} = A_{ji}$. Then,* $\det A^t = \det A$.

Proof.

$$\begin{aligned}
\det A^t &= \sum_{\sigma \in S_n} A^t_{1\sigma(1)} \cdots A^t_{n\sigma(n)} \operatorname{sgn} \sigma \\
&= \sum_{\sigma \in S_n} A_{\sigma(1)1} \cdots A_{\sigma(n)n} \operatorname{sgn} \sigma \\
&= \sum_{\sigma \in S_n} A_{1\sigma^{-1}(1)} \cdots A_{n\sigma^{-1}(n)} \operatorname{sgn} \sigma.
\end{aligned}$$

Now, the result follows because $\operatorname{sgn} \sigma^{-1} = \operatorname{sgn} \sigma$. /////

Let $\widehat{A_{ij}}$ be the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i^{th} row and j^{th} column. $(-1)^{i+j} \det \widehat{A_{ij}}$ is called the $(i,j)^{th}$ *cofactor of A*.

Proposition 6. *For any j:*

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \det \widehat{A_{ij}}. \tag{$*$}$$

Remark 1. Let us look at the $n = 2$, $j = 1$ case. Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\widehat{A_{11}} = d$ and $\widehat{A_{21}} = b$, and

$$\sum_{i=1}^{2} (-1)^{i+1} A_{i1} \det \widehat{A_{i1}} = ad - cb.$$

Community Problem 1. Suppose $f : V^k \to \mathbb{F}$ is multilinear and $f(\mathbf{w}) = 0$ whenever a pair of *adjacent* entries in $\mathbf{w}$ are equal. Prove that f is alternating.

Proof of Proposition 6. Fix n and j. Suppose A is the identity matrix. Then in the right hand side of $(*)$ the i^{th} term is non-zero only when $i = j$. Since $\widehat{A_{jj}}$ is the identity matrix, this term is equal to 1. We finish the proof in two steps, showing first the each term on the right hand side is an n-multlinear functional of the rows to conclude that entire sum is n-multilinear, and second—using the "Community Problem"— that the sum is alternating. Now, for any fixed i, $\det \widehat{A_{ij}}$ regarded as a functional on the rows of A other than the i^{th} is $n-1$-multilinear. Moreover, A_{ij} is linear on the i^{th} row. Thus the product, $A_{ij} \det \widehat{A_{ij}}$ is n-multilinear. (The reader should check this carefully. Notice that if $g_i : W_i \to \mathbb{F}$ is linear for each $i = 1, \ldots, \ell$, then the map $(w_1, w_2, \ldots, w_\ell) \mapsto g_1(w_1) \cdot g_2(w_2) \cdots g_\ell(w_\ell)$ from $\prod_{i=1}^{\ell} W_i$ to $\mathbb{F}$ is ℓ-multilinear.) It remains only to see that the right hand side of $(*)$ is alternating, and it suffices to show that it vanishes if two consecutive rows of A are equal. Suppose rows k and $k+1$ are equal. In the RHS of $(*)$, every term but those indexed by $i = k$ and $i = k+1$ vanish, because when $i \notin \{k, k+1\}$, $\widehat{A_{ij}}$ has a repeated row. So, we need to show:

$$(-1)^{k+j} A_{kj} \det \widehat{A_{kj}} + (-1)^{(k+1)+j} A_{(k+1)j} \det \widehat{A_{(k+1)j}} = 0,$$

i.e.,

$$\det \widehat{A_{kj}} = \det \widehat{A_{(k+1)j}},$$

and this is obvious. /////

Proposition 7. *(Vandermonde matrix). Let $r_1, \ldots, r_n$ be scalars and let A be the matrix $A_{ij} = r_j^{i-1}$. Then*

$$\det A = \prod_{j>i} (r_j - r_i).$$

Whole Community Problem. Find (on your own, or using references) as many proofs of this as you can.

Eigenvalues and eigenvectors

Definition. Let A be an $n \times n$ matrix. Then $v \in \mathbb{F}^n \setminus \{\overline{0}\}$ is called an *eignvector* for A with *eigenvalue* λ if $Av = \lambda v$ for some scalar λ.

Proposition. *A has an eigenvector with eigenvalue λ if and only if* $\det(\lambda I - A) = 0$.

Comment. $\det(\lambda I - A)$ is a polynomial of degree n in λ. It is called the *characteristic polynomial* of A. We find the eigenvalues of A by finding the roots of this polynomial. We find the *eigenspace for λ* by finding the null space of $\lambda I - A$.

Proposition. *Suppose $v_1, \ldots, v_k$ are eigenvectors for A (an $n \times n$ matrix) with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Then, $v_1, \ldots, v_k$ are linearly independent.*

Proof. Suppose $c_1 v_1 + \cdots + c_k v_k = 0$. We show that all the c_i vanish. Multiplying by A repeatedly, we get $\lambda_1^j c_1 v_1 + \cdots + \lambda_k^j c_k v_k = 0$ for $j = 1, \ldots, k-1$. Let Λ be the Vandermonde matrix for the λ_i—which is non-singular—and let B be the matrix with rows $c_i v_i$. Then, $\Lambda B = 0$, so all the rows of B vanish, so all the c_i vanish. /////

Homework. 19, 20, 23, 26, 28, 32, 33 (Due Friday, Sept 21.)

Community Problems. 31, 34, 40, 41, 42, 43, 44