**M7210 Lecture 12.** Wednesday September 19, 2012

Monday's class was cancelled due to a bomb scare that closed down the whole university. This reminds me of a joke: *A bull swallowed a bomb. What do you call it?* (Abominable.) *The bomb exploded. What do you call it?* (Noble.)

## Inner Product Spaces

Up to this point, we have been looking at abstract vector spaces. Bases have been introduced in order to create notation for elements or maps, but no basis or set of bases has been given any preference over any other.[1] Working at this level of abstraction, we have been able to describe and prove theorems about many important constructions: sub-spaces, dimension, quotients, products, duality, determinants and eigenvectors.

We are missing a way to measure the lengths of vectors or the angles between them. Lacking this, the vector spaces we have been looking at have minimal geometric structure. To make this point clear, consider a 1-dimensional vector space $V$ over $\mathbb{R}$. Any non-zero element of $V$ could be used as a unit of length, but we can attribute lengths to the elements of $V$ only after a unit length has been chosen. If no unit is distinguished and preserved, then there is a linear automorphism of $V$ carrying any non-zero element to any other. Having chosen a unit, on the other hand, there are only two linear automorphisms that preserve its length. In a 2-dimensional space, we need more than just a unit of length, we also need a way to measure vectors that are not multiples of the unit, and things become much more interesting.

It turns out that if we equip a vector space with an *inner product*, we can make sense of lengths and angles. A vector space endowed with this additional structure is a much richer object. It particular, such a vector space has fewer symmetries than a "naked" vector space because they are required to preserve more structure. There are enough symmetries left to be interesting, yet not so many as to become boring.

In this lecture and the next two, we will assume $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$.

## Inner Products

**Definition.** (Real case.) Let $V$ be a vector space over $\mathbb{R}$. An *inner product on* $V$ is a bilinear form[2]

$$(u, v) \mapsto \langle u, v\rangle : V \times V \to \mathbb{R},$$

such that:
a) $\langle \cdot, \cdot\rangle$ is symmetric—i.e., $\langle u, v\rangle = \langle v, u\rangle$ for all $u, v \in V$,
b) $\langle v, v\rangle \geq 0$ for all $v \in V$, and
c) for all $v \in V$, $\langle v, v\rangle = 0$ only if $v = 0$.
A *real inner-product space* is a vector space over $\mathbb{R}$ equipped with an inner product.

**Definition.** (Complex case.) Let $V$ be a vector space over $\mathbb{C}$. A function $f : V \to \mathbb{C}$ is said to be *congugate linear* if it is additive—$f(u + v) = f(u) + f(v)$—and satisfies $f(cv) = \overline{c}f(v)$, where $\overline{a + ib} := a - ib$. A *Hermitian inner product on* $V$ is a function

$$(u, v) \mapsto \langle u, v\rangle : V \times V \to \mathbb{C},$$

that is linear in the first variable and conjugate linear in the second, and which in addition satisfies:
a) $\langle \cdot, \cdot\rangle$ is *Hermitian symmetric*—i.e., $\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all $u, v \in V$,
b) $\langle v, v\rangle \geq 0$ for all $v \in V$, and
c) for all $v \in V$, $\langle v, v\rangle = 0$ only if $v = 0$.
A *Hermitian inner-product space* is a vector space over $\mathbb{C}$ equipped with a Hermitian inner product.

[1] $\mathbb{F}^n$ and its distinguished "standard" base is an exception. $\mathbb{F}^n$ is not merely a vector space, but a much more complex object—unless, of course, we systematically ignore the standard basis!

[2] "Form" is a synonym for "functional".

If we refer to an "inner-product space over $\mathbb{F}$," we mean a real or Hermitian space, depending on whether $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

*Examples 1.*

- In $\mathbb{R}^n$, the *dot product*—defined by $(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) = a_1 b_1 + \cdots a_n b_n$—is an inner product. Note that the definition depends on choice of basis. Given any $n$-dimensional real vector space $V$ with basis $E$, we could define an inner product relative to $E$ by

$$\langle u, v \rangle_E := \begin{pmatrix} u \\ E \end{pmatrix} \cdot \begin{pmatrix} v \\ E \end{pmatrix}.$$

- In $\mathbb{C}^n$, we may define an Hermitian inner product by $\langle x, y \rangle := x_1 \overline{y_1} + \cdots + x_n \overline{y_n}$. The form of the definition ensures that $\langle v, v \rangle$ is a non-negative real, so it has a square root.

**Norms**

**Definition.** Let $V$ be an inner-product space. The *norm* of $v \in V$ is $||v|| := \sqrt{\langle v, v \rangle}$.

*Example 2.*

$$\begin{aligned} ||u+v||^2 &= \langle u+v, u+v \rangle \\ &= \langle u, u+v \rangle + \langle u, u+v \rangle \\ &= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\ &= \langle u, u \rangle + \langle u, v \rangle + \overline{\langle u, v \rangle} + \langle v, v \rangle \\ &= ||u||^2 + 2\mathrm{Re}\,\langle u, v \rangle + ||v||^2 \end{aligned}$$

This computation is written for $\mathbb{F} = \mathbb{C}$. It can be read for $\mathbb{F} = \mathbb{R}$ by simply ignoring the over-line and the Re.

*Example 3.* Suppose $\omega = e^{i\theta} \in \mathbb{C}$. Then $|\langle \omega u, v \rangle| = |\omega \langle u, v \rangle| = |\omega| \cdot |\langle u, v \rangle| = |\langle u, v \rangle|$.

**Proposition 1.** *(Schwartz inequality). In any inner-product space,*

$$|\langle u, v \rangle| \leq ||u||\,||v|| \text{ for all } u, v \in V.$$

For the proof, see book. I will comment on this later. The proof given by the book at the present stage of exposition appears as if "pulling a rabbit out of a hat." We will return to it later to provide the missing motivation.

**Proposition 2.** *Let $V$ be an inner-product space $V$. Then:*

*a) For all $v \in V$, $||v|| \geq 0$, and $||v|| = 0$ if and only if $v = 0$.*
*b) For all $v \in V$ and all scalars $c$, $||cv|| = |c|||v||$;*
*c) For all $u, v \in V$, $||u+v|| \leq ||u|| + ||v||$ (Triangle Inequality);*
*d) For all $u, v \in V$, $||u+v||^2 + ||u-v||^2 = 2||u||^2 + 2||v||^2$ (Parallelogram Law).*

*Proof.* a) and b) are left to you to check. For c),

$$\begin{aligned} ||u+v||^2 &= ||u||^2 + 2\mathrm{Re}\,\langle u, v \rangle + ||v||^2, \quad \text{by Example 2,} \\ &\leq ||u||^2 + 2|\langle u, v \rangle| + ||v||^2, \quad \text{since } \mathrm{Re}\, z \leq |z|, \\ &\leq ||u||^2 + 2||u||\,||v|| + ||v||^2, \quad \text{by Schwartz,} \\ &= \big(||u|| + ||v||\big)^2. \end{aligned}$$

Now take square roots of both sides. Part d), is immediate from the computation in Example 2. /////

*Remark.* By the Triangle Inequality, $||u+v|| \leq ||2u|| + ||v-u||$, so

$$||u+v|| - ||u-v|| \leq ||2u||. \tag{$*$}$$