**M7210 Project** Wednesday September 19, 2012

In this project, you will reproduce the proof of the theorem stated below, which is from

- Jordan, P. & von Neumann, J., "On Inner Products in Linear, Metric Spaces," *Annals of Mathematics*, 36 (1935), 719–723.

This paper a supplement to the preceding paper by M. Frechet in the same issue of the *Annals*, which contains an axiomatic treatment of inner products and norms.

**Theorem.** *Let $V$ be a vector space. Suppose $||\cdot|| : V \to \mathbb{F}$ is a function that satisfies conditions a) through d) in Proposition 2.*

a) *Suppose $\mathbb{F} = \mathbb{R}$. Define $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ by $\langle u, v\rangle := \frac{1}{4}\big(||u+v||^2 - ||u-v||^2\big)$. Then this is a real inner product on $V$.*

b) *Suppose $\mathbb{F} = \mathbb{C}$. Define $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ by $\langle u, v\rangle := \frac{1}{4}\big(||u+v||^2 + i|u+iv||^2 - ||u-v||^2 - i||u-iv||^2\big)$. Then this is an Hermitian inner product on $V$.*

*Guided proof*

**a)** Fill in the details in the following argument to prove part a):

i) It suffices to show that $\langle\cdot,\cdot\rangle$ is linear in the first variable. Explain why.

ii) Let $u_1, u_2, v \in V$. Show that

$$\langle u_1 + u_2, v\rangle + \langle u_1 - u_2, v\rangle = 2\langle u_1, v\rangle. \tag{1}$$

iii) Using (1), show that for ant $s, t, v \in V$:

$$\langle s, v\rangle + \langle t, v\rangle = \langle s+t, v\rangle. \tag{2}$$

iv) Using (2), show $\langle qu, v\rangle = q\langle u, v\rangle$ for any $q \in \mathbb{Q}$.

v) Using the definition of $\langle u, v\rangle$, show that if $u, v \in V$ are given then there is a constant $K(u, v)$ (depending on $u$ and $v$) such that

$$|\langle c\,u, v\rangle| \le c\,K(u, v) \quad \text{for all } 0 < c \le 1.$$

vi) Assume $r \in \mathbb{R}$. Show that there is a constat $C$ such that

$$|r\langle u, v\rangle - \langle ru, v\rangle| \le C/i \quad \text{for all } i = 1, 2, \ldots.$$

(Hint: Select a sequence $q_i$ of rationals such that $|r - q_i| < 1/i$, and use the triangle inequality.)

vii) Conclude that $r\langle u, v\rangle = \langle r\,u, v\rangle$, and thus finish the proof of part a).

**b)** Using or extending the ideas in Part a), prove part b).